

MULTISTRING VERTICES AND
HYPERBOLIC KAC MOODY ALGEBRAS**R. W. Gebert[†], H. Nicolai[‡],*IInd Institute for Theoretical Physics, University of Hamburg
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Multistring vertices and the overlap identities which they satisfy are exploited to understand properties of hyperbolic Kac Moody algebras, and E_{10} in particular. Since any such algebra can be embedded in the larger Lie algebra of physical states of an associated completely compactified subcritical bosonic string, one can in principle determine the root spaces by analyzing which (positive norm) physical states decouple from the N -string vertex. Consequently, the Lie algebra of physical states decomposes into a direct sum of the hyperbolic algebra and the space of decoupled states. Both these spaces contain transversal and longitudinal states. Longitudinal decoupling holds generally, and may also be valid for uncompactified strings, with possible consequences for Liouville theory; the identification of the decoupled states simply amounts to finding the zeroes of certain “decoupling polynomials”. This is not the case for transversal decoupling, which crucially depends on special properties of the root lattice, as we explicitly demonstrate for a non-trivial root space of E_{10} . Because the N -vertices of the compactified string contain the complete information about decoupling, all the properties of the hyperbolic algebra are encoded into them. In view of the integer grading of hyperbolic algebras such as E_{10} by the level, these algebras can be interpreted as interacting strings moving on the respective group manifolds associated with the underlying finite-dimensional Lie algebras.

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1 Introduction

This paper brings together two lines of development both of which are intimately connected to core issues of modern string theory, namely the theory of indefinite and, more specifically, hyperbolic Kac Moody algebras on the one hand, and (aspects of) string field theory on the other. Little is known about the infinite-dimensional Lie algebras and groups that have been suggested as candidates for a unified symmetry of (super)string theory; similarly, it is far from assured that we are already in possession of the proper physical concepts necessary to deal with Planck scale physics and quantum gravity. Nonetheless, the developments of the past decade have taught us that progress in one of these areas may well depend on advances in the other. The main purpose of the present article is to show how methods developed in the early days of string theory may provide a better understanding of the mathematical structures encountered in the theory of indefinite Kac Moody algebras. More specifically, we will work out the connection between hyperbolic Kac Moody algebras and multistring vertices, and in particular make precise the sense in which the structure constants of such algebras can be identified with the scattering amplitudes for the physical states corresponding to the root space elements of the Lie algebra (although this seems to be almost a folklore result in connection with “monstrous” Lie algebras based on the lattice $\mathbb{H}_{25,1}$, the point has never been made for hyperbolic Kac Moody algebras to the best of our knowledge). In fact, we shall argue (and offer concrete evidence) that the multistring vertices of a suitably compactified bosonic string “know everything” there is to know about certain hyperbolic Kac Moody algebras such as E_{10} .

N -string vertices played an important role in the early history of string theory following the discovery of the N -tachyon scattering amplitude [61, 39]. While scattering amplitudes were originally computed by evaluating products of suitable vertex operators (such as $:\exp(i\mathbf{v}\cdot\mathbf{X}):$ for the tachyon) between two vacuum states in a one-string Fock space [22] (for a review see [1]), it was soon realized that they could alternatively be obtained by attaching the physical states involved in the scattering to the legs of a multistring vertex acting on a multistring Fock space. The beginning of this development was the discovery of a special three-vertex with the requisite properties in [57]; a slightly more symmetric, but physically equivalent three-vertex was found shortly after in [9]. The problem of constructing an N -vertex for arbitrary $N > 3$ was solved in [41, 54] by sewing $N - 2$ three-vertices. Not only did these N -vertices describe N -string scattering, but they provided transparent proofs of various properties of the scattering amplitudes, such as duality. The formalism did, however, have some drawbacks: the N string vertices obtained by sewing were rather complicated and their properties were not at all apparent. For instance, the action of the Virasoro generators on the N -vertex was not properly understood; furthermore, the investigation of loop effects in this framework led to incorrect results as negative norm states were found to propagate within the loops.

A new approach to multistring vertices was initiated in 1986 [48, 47, 49] and further developed in [51, 50, 63]. It was shown there that N -vertices could be directly and very explicitly determined without recourse to sewing from a simple set of defining equations called overlap equations. These come in two varieties, namely as unintegrated and integrated overlaps; although the latter contain somewhat less information than the unintegrated overlaps they are frequently more useful for practical calculations as we will also see in this paper. An especially appealing feature of this formalism is the elegant geometrical interpretation of the overlap equations in terms of coordinate patches associated with every string emitted from the worldsheet; the N -vertex is, in fact, completely characterized by the transition functions between these patches. One finds in this way an infinite class of vertices corresponding to different transition functions, which are physically equivalent but differ off-shell. As an added bonus, general properties of the multistring vertices could now be fully elucidated. Furthermore, the multistring vertex for arbitrary genus was found in [51] and ghosts were introduced in [48, 17]. The first of these papers on ghost vertices enabled the Copenhagen group to refine and complete the original sewing program [14, 59] and to derive a particularly elegant expression for the bosonic loop measure [13]. The integrated overlap equations, which can be obtained from the overlap equations by contour integration, were independently studied and extensively applied in the context of the so-called Grassmannian approach to string theory [2]. For an account of the various operator approaches and further references, see also [63].

Despite the well known and close links between string theory and Kac Moody algebras [3, 58, 20, 32] (see also [37], which is the standard textbook on the subject), it appears that multistring technology has not really been exploited for its full worth in the Lie algebra context, as much of the pertinent literature deals solely with one-string Fock spaces and one-string vertex operators. We will here demonstrate that multistring vertices may serve as valuable tools for the analysis of the multiple commutators arising in the theory of hyperbolic Kac Moody algebras, and that they offer new and promising insights into their structure, which has so far defied all attempts at a complete understanding. Multistring vertices can in principle provide exhaustive information about the root spaces of such algebras and may thus bring us closer to the ultimate goal of finding a manageable representation of the Lie algebra elements that would be analogous to the current algebra representation for affine algebras, a task which is even harder than the calculation of root multiplicities. To appreciate the difficulties readers should recall that there is so far not a single example of a hyperbolic Kac Moody algebra for which even the root multiplicities are completely known. We will in this paper mainly rely on the results of [26], where a DDF type approach adapted to the root lattice was developed. The crucial result invoked there is that any hyperbolic Kac Moody algebra $\mathfrak{g}(A)$ can be embedded in a larger Lie algebra of physical states \mathfrak{g}_Λ [5], where Λ is the root lattice associated with the Cartan matrix A . Consequently, the elements of $\mathfrak{g}(A)$ can be completely characterized as the orthogonal complement in \mathfrak{g}_Λ of those states that cannot be reached by multiple commutators of the basic Chevalley Serre generators. Although the idea of defining the Kac Moody algebra by what is *not* in it rather than by what is in it seems completely counterintuitive, it turns out, somewhat surprisingly, that these “missing” states (mostly referred to as “decoupled states” in this paper) are easier to identify than the Lie algebra elements themselves if astute use is made of multistring vertices: one must only check that they decouple from the relevant N -vertex! We emphasize that the decoupling we are concerned with here is of a novel type, and not like the decoupling of null states in $d = 26$ string theory, in that the decoupled states have *positive* norm (this casts some doubt on the existence of a BRST-type cohomology, which would explain decoupling). As in [26], our principal example will be the maximally extended hyperbolic Kac Moody algebra E_{10} , even though many of our results are valid more generally.

Our results also suggest a new interpretation of hyperbolic Kac Moody algebras in physics: they describe a theory of interacting strings at tree level, where the usual one-string Fock space is replaced by the basic representation of the underlying affine Kac Moody subalgebra. The basic representation in turn is nothing but the set of purely transversal states built on the tachyonic groundstate $|\mathbf{r}_{-1}\rangle$ and its orbit under the action of the affine Weyl group; here \mathbf{r}_{-1} is the over-extended root, and the tachyonic momenta that can be generated by the affine Weyl group are of the form $\mathbf{v} = \mathbf{r}_{-1} + (\frac{1}{2}\mathbf{b}^2)\boldsymbol{\delta} + \mathbf{b}$, where $\boldsymbol{\delta}$ is the affine null root, and \mathbf{b} has support on the finite subdiagram of the full Dynkin diagram obtained by deleting the extended and overextended nodes. We note that our results in some sense constitute a realization of a remark in [64], where it was proposed to interpret E_{10} as a symmetry acting on a multistring Fock space. As pointed out there, the group corresponding to E_{10} would admit $E_9 \times U(1)$ as a subgroup, where the $U(1)$ factor is generated by the central charge c (it is hard to imagine what the group would look like if one does not even know the Lie algebra; see, however, [60] for some further information on this topic). Accordingly, the central charge c , whose integer eigenvalues are called the “levels” of the relevant representations of the affine subalgebra, should be viewed as an operator counting the number of strings. If we accept this interpretation, the following picture emerges. As shown in [27], the allowed one-string Fock spaces of a bosonic string, the “internal” part of which propagates on a Lie group manifold, constitute integrable highest-weight representations of the associated affine algebra. But since

the basic representation is the simplest of these we can think of the hyperbolic algebra as a multistring Fock space for interacting strings moving on the group manifold corresponding to the underlying finite Lie algebra (in which case the target space contains no uncompactified spacetime part). The Chevalley involution θ (cf. Eq. (2.96)) would then play the role of a charge conjugation operator, exchanging strings and “antistrings” (i.e. representations and contragredient representations). Note that we employ two different string compactifications. We start with a (subcritical) string which is completely compactified on a Minkowski torus and into whose physical state space the hyperbolic Kac Moody algebra is embedded as a subalgebra; this model is merely needed in order to have an explicit realization of the algebra and has no immediate physical interpretation. Rather, the graded structure of the hyperbolic algebra is tied to the interacting string picture where now the strings live on a Lie group manifold.

For evident reasons the hyperbolic algebra E_{10} has also been advanced as a candidate symmetry of the superstring in ten dimensions. Despite the considerable efforts invested into the search for a proper formulation of string field theory, however, it has not been possible so far to exhibit a multistring Hamiltonian (or, rather, a Wheeler-DeWitt type constraint operator) for the superstring in ten dimensions which would admit such a symmetry and commute with it. Besides, one would naively expect the relevant symmetry algebra to be a superalgebra, not a bosonic Lie algebra. For a superalgebra, one must take into account additional roots of length one on the root lattice for the fermionic generators [32], but enlarging the even selfdual E_{10} root lattice $\Pi_{9,1}$ in this way would put a stain on its beauty in our opinion. To construct a superalgebra one could conceivably compactify the string on the unique odd selfdual Lorentzian lattice $I_{9,1} \equiv \mathbf{Z}^{9,1}$; see [6] for some comments and speculations. But since $I_{9,1}$ does not contain $\Pi_{9,1}$ as a sublattice, E_{10} could certainly not emerge as a bosonic subalgebra in such a construction. A related question is whether there exists a (different?) superalgebra which can be embedded into the space of bosonic and fermionic physical states of the fully compactified superstring in the same way as E_{10} is embedded into the space of physical states of a bosonic string. Curiously, such an algebra would have no real roots at all due to the absence of tachyonic states in the superstring. Hence, neither superalgebra would deserve to be called “super- E_{10} ” because neither would contain E_{10} as a bosonic subalgebra¹! Therefore, a more attractive possibility from our point of view is that there is actually no need for a superalgebra because E_{10} may already be secretly “aware of” supersymmetry even though it is a bosonic symmetry by all appearances. That this is not an entirely far-fetched idea was explicitly shown in the context of two-dimensional Poincaré supergravities, whose local supersymmetry transformations can be “bosonized” into certain Kac Moody gauge transformations in the associated linear systems [53, 52]. It would, however, mean that the theory appropriate for E_{10} is not the superstring, but some other theory (perhaps involving supergravity in eleven dimensions [11]).

The implications of our results for string field theory and the possible relation to quantum Liouville theory remain fascinating topics for future research. Namely, the decoupling of certain positive norm physical states holds not only on the lattice, but also in the continuum. This raises the question of whether systematically discarding these states can lead to a consistent quantum theory in the “forbidden zone” $1 < d < 25$. Furthermore, we suspect that radiative (string loop) corrections will also have a role to play in the context of hyperbolic Kac Moody algebras, although at our present level of understanding it is not at all clear where they might enter. In particular, it would be necessary to find out whether these corrections are compatible with our decoupling mechanism; in this case, the vertices with loops will almost certainly contain information about the E_{10} states at arbitrary level, as the latter would propagate inside loops. The multistring description also suggests a subtle link between the worldsheet topology (i.e. its genus and punctures, see [44, 28] for a nice description) and the level which grades the hyperbolic algebra, indicating that hyperbolic symmetries might naturally involve all orders of string perturbation theory.

Let us now briefly summarize the content of this paper.

Section 2 is devoted to an exposition of the one-string model. We have decided to work within the modern framework of vertex algebras since it makes the Lie algebra structure of the space of physical string states especially transparent. After a short review of vertex algebras we shall present in detail the string model relevant for the paper, i.e. the closed bosonic in $d < 26$ dimensions with *all* target space coordinates toroidally compactified. In particular, the relation of the Lie algebra of physical states and the embedded Kac Moody algebra will be worked out. The difference between the string scalar product and the invariant bilinear form for the Lie algebra will be explained.

In Sect. 3 we will deal with the multistring formalism. Starting from the operatorial definition of N -string vertices via overlap identities it will be shown how to obtain their explicit form in the usual oscillator representation. Then the conformal transformation properties of the vertices will be derived.

The content of the first two sections will be brought together in Sect. 4 where we set up a correspondence between

¹It is known that the *finite*-dimensional exceptional Lie algebras of E -type possess no superextensions [36].

one-string vertex operators and three-vertices taking into account some additional physical assumptions. Then the machinery of sewing three-vertices is presented. With this method at hand we subsequently discuss the notions of locality and duality for the four-vertex and their generalizations to N -vertices. Finally two specific three-vertices are explicitly constructed, one of which leading to a vertex operator that satisfies all the axioms of a vertex algebra. In particular, we find a rather simple, diagrammatical proof of the Jacobi identity. Moreover, we establish the equivalence of the overlap identities for this three-vertex and the so-called Jacobi identity for intertwining operators.

Section 5 contains the core results of this paper. Making extensive use of the overlap equations, we discuss the decoupling of longitudinal and transversal states in detail, and show how root spaces can be determined without explicit computation of commutators. On the way, structure constants for N -fold commutators will be introduced and it will be explained in which sense the structure constants of hyperbolic algebras can be identified with the scattering amplitudes for the underlying compactified string. These issues will be addressed in Sect. 5.1, where we describe in detail how to rewrite (multiple) commutators in terms of N -string vertices. Section 5.2 summarizes the essential results of [26] needed here and describes how to recover them in the multistring formalism. Section 5.3 is devoted to an analysis of the longitudinal decoupling; there we will introduce the “decoupling polynomials” from which one can directly read off which longitudinal states couple and which decouple, and compute these polynomials up to degree seven. These results go considerably further than the ones previously obtained. In Sect. 5.4 we deal with the decoupling of transversal states. As this is the least understood part of our construction we will make no attempt at a systematic treatment but rather illustrate the phenomenon in terms of a concrete (and non-trivial) example.

2 The vertex algebra associated with the compactified string

We shall study a chiral sector of a closed bosonic string living on a Minkowski torus as spacetime, i.e. we compactify all target space coordinates. Consequently, the center of mass momenta of the string form a lattice with Minkowski signature. It turns out that the states and vertex operators realize a mathematical structure called vertex algebra [5]. The latter framework will be convenient to establish the connection between string vertices and multiple commutators. In our exposition we will closely follow the review [23].

2.1 Vertex algebras

In the vertex algebra formalism one works with *formal* variables z, w, y, \dots and *formal* Laurent series. For a vector space S , say, we set $S[[z, z^{-1}]] = \{\sum_{n \in \mathbf{Z}} s_n z^{-n-1} | s_n \in S\}$. For a formal series we use the following residue notation:

$$\text{Res}_z \left[\sum_{n \in \mathbf{Z}} s_n z^{-n-1} \right] = s_0, \quad (2.1)$$

so that we may think of $\text{Res}_z [\dots]$ as the formal expression for the operation $\oint_0 \frac{dz}{2\pi i} [\dots]$ in complex analysis. We shall also need the analogue of the δ function which is defined as $\delta(z) := \sum_{n \in \mathbf{Z}} z^n$. Formally, this is the Laurent expansion of the classical δ function at $z = 1$ and indeed it enjoys similar properties. Formal calculus is presented in great detail in [21].

Definition 1

A **vertex algebra** is a \mathbf{Z} -graded vector space

$$\mathcal{F} = \bigoplus_{n \in \mathbf{Z}} \mathcal{F}^n, \quad (2.2)$$

equipped with a linear map $\mathcal{V} : \mathcal{F} \rightarrow (\text{End } \mathcal{F})[[z, z^{-1}]]$, which assigns to each state $\psi \in \mathcal{F}$ a **vertex operator** $\mathcal{V}(\psi, z)$, and the vertex operators satisfy the following axioms:

1. **(Regularity)** If $\psi, \varphi \in \mathcal{F}$ then

$$\text{Res}_z [z^n \mathcal{V}(\psi, z) \varphi] = 0 \quad \text{for } n \text{ sufficiently large} \quad (2.3)$$

and n depending on ψ and φ .

2. **(Vacuum)** *There is a preferred state $\mathbf{1} \in \mathcal{F}$, called the vacuum, satisfying*

$$\mathcal{V}(\mathbf{1}, z) = \text{id}_{\mathcal{F}}. \quad (2.4)$$

3. **(Injectivity)** *There is a one-to-one correspondence between states and vertex operators:*

$$\mathcal{V}(\psi, z) = 0 \iff \psi = 0. \quad (2.5)$$

4. **(Conformal vector)** *There is a preferred state $\omega \in \mathcal{F}$, called the conformal vector, such that its vertex operator,*

$$\mathcal{T}(z) \equiv \mathcal{V}(\omega, z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2}, \quad (2.6)$$

- (a) *gives the **Virasoro algebra** with some central charge $c \in \mathbf{R}$,*

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}; \quad (2.7)$$

- (b) *provides a **translation generator**, L_{-1} ,*

$$\mathcal{V}(L_{-1}\psi, z) = \frac{d}{dz}\mathcal{V}(\psi, z) \quad \text{for every } \psi \in \mathcal{F}; \quad (2.8)$$

- (c) *gives the grading of \mathcal{F} via the eigenvalues of L_0 ,*

$$L_0\psi = n\psi \equiv h_\psi \psi \quad \text{for every } \psi \in \mathcal{F}^n, n \in \mathbf{Z}; \quad (2.9)$$

*the eigenvalue h_ψ is called the **(conformal) weight** of ψ .*

5. **(Jacobi identity)** *For every $\psi, \varphi \in \mathcal{F}$,*

$$\begin{aligned} y^{-1}\delta\left(\frac{z-w}{y}\right)\mathcal{V}(\psi, z)\mathcal{V}(\varphi, w) - y^{-1}\delta\left(\frac{-w+z}{y}\right)\mathcal{V}(\varphi, w)\mathcal{V}(\psi, z) \\ = w^{-1}\delta\left(\frac{z-y}{w}\right)\mathcal{V}(\mathcal{V}(\psi, y)\varphi, w), \end{aligned} \quad (2.10)$$

where binomial expressions have to be expanded in nonnegative integral powers of the second variable.

Since vertex operators are operator-valued formal Laurent series, we can give an alternative formulation (see e.g. [5]) of the axioms of a vertex algebra using the mode expansion

$$\mathcal{V}(\psi, z) = \sum_{n \in \mathbf{Z}} \psi_n z^{-n-1}. \quad (2.11)$$

From the axioms one can derive some important properties of vertex algebras. For example, the Jacobi identity implies the relations

$$[L_{-1}, \mathcal{V}(\psi, z)] = \frac{d}{dz}\mathcal{V}(\psi, z), \quad (2.12)$$

$$[L_0, \mathcal{V}(\psi, z)] = \left(z\frac{d}{dz} + h_\psi\right)\mathcal{V}(\psi, z) \quad \text{if } \psi \in \mathcal{F}^{h_\psi}, \quad (2.13)$$

which give, respectively,

(Translation property)

$$e^{xL_{-1}}\mathcal{V}(\psi, z)e^{-xL_{-1}} = \mathcal{V}(\psi, z+x), \quad (2.14)$$

(Scaling property)

$$e^{xL_0}\mathcal{V}(\psi, z)e^{-xL_0} = e^{xh_\psi}\mathcal{V}(\psi, e^x z) \quad \text{if } \psi \in \mathcal{F}^{h_\psi}, \quad (2.15)$$

for every $x \in w\mathbf{C}[[w]]$. Thus L_{-1} and L_0 generate translations and scale transformations, respectively.

Combining injectivity with the Jacobi identity one arrives at

(Skew symmetry)

$$\mathcal{V}(\psi, z)\varphi = e^{zL_{-1}}\mathcal{V}(\varphi, -z)\psi, \quad (2.16)$$

which shows that the vertex operator $\mathcal{V}(\psi, z)$, when applied to the vacuum, “creates” the state $\psi \in \mathcal{F}$:

$$\mathcal{V}(\psi, z)\mathbf{1} = e^{zL_{-1}}\psi. \quad (2.17)$$

We shall denote by $\mathcal{P}^h \subset \mathcal{F}^h$ the space of **(conformal) highest weight vectors** or **primary states** satisfying

$$L_0\psi = h\psi, \quad (2.18)$$

$$L_n\psi = 0 \quad \forall n > 0. \quad (2.19)$$

For example, in any vertex algebra the vacuum is a primary state of weight 0 and therefore \mathcal{F}^0 is always at least one-dimensional. We can deduce from the Jacobi identity that, for $\psi \in \mathcal{P}^h$,

$$[L_n, \mathcal{V}(\psi, z)] = z^n \left\{ z \frac{d}{dz} + (n+1)h \right\} \mathcal{V}(\psi, z) \quad \forall n \in \mathbf{Z}, \quad (2.20)$$

or, in terms of mode operators,

$$[L_n, \psi_m] = \{(h-1)(n+1) - m\}\psi_{m+n}, \quad (2.21)$$

i.e. $\mathcal{V}(\psi, z)$ is a so-called **(conformal) primary field** of weight h . From now on we shall use the notation $\Phi(z)$ for a primary field. Upon exponentiation we find that

(Projective change)

$$e^{xL_n}\Phi(z)e^{-xL_n} = \left(\frac{\partial p(z)}{\partial z} \right)^h \Phi(p(z)) \quad \forall n \neq 0, \quad (2.22)$$

for every $x \in w\mathbf{C}[[w]]$ where $p(z) = z(1 - nxz^n)^{-1/n}$.

Since the operators $\{L_{-1}, L_0, L_1\}$ satisfy the **su(1, 1)** Lie algebra

$$[L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0, \quad (2.23)$$

we have the following Möbius transformation properties of the vertex operators (see also [31]): If $\psi \in \mathcal{F}$ is a **quasiprimary state** of weight h , i.e. ψ satisfies $L_0\psi = h\psi$ and $L_1\psi = 0$, then

$$\hat{\mathcal{M}}\mathcal{V}(\psi, z)\hat{\mathcal{M}}^{-1} = \left[\frac{d\mathcal{M}(z)}{dz} \right]^h \mathcal{V}(\psi, \mathcal{M}(z)), \quad (2.24)$$

where

$$\mathcal{M}(z) := \frac{az+b}{cz+d}, \quad \hat{\mathcal{M}} := e^{\frac{b}{d}L_{-1}} \left(\frac{\sqrt{ad-bc}}{d} \right)^{2L_0} e^{-\frac{c}{d}L_1}, \quad (2.25)$$

for $a, b, c, d \in w\mathbf{C}[[w]]$.

For our purposes it will be very useful to have a relation between the Jacobi identity for vertex algebras and the notions of locality and duality from conformal field theory. This can be achieved by considering matrix elements of products of vertex operators w.r.t. some nondegenerate bilinear form $\langle - | - \rangle$. It follows from the axioms that any matrix element of a vertex operator is a Laurent *polynomial* in z ,

$$\langle \chi | \mathcal{V}(\psi, z)\varphi \rangle \in \mathbf{C}[z, z^{-1}] \quad \forall \chi, \psi, \varphi \in \mathcal{F}, \quad (2.26)$$

so that these three-point correlation functions may be regarded as meromorphic functions of z . Of course, we formally identify χ with an “out-state” inserted at $z = \infty$ and φ with an “in-state” inserted at $z = 0$.

We recall that in the Jacobi identity axiom (2.10) we used the terminology ‘expansion in nonnegative integral powers of the second variable’. We have to make this convention mathematically more precise. Let $\mathbf{C}[z, w]_S$ denote the subring of the field of rational functions which can be written as

$$f(z, w) = \frac{g(z, w)}{w^s \prod_{l=1}^r (a_l z + b_l w)} \quad (2.27)$$

where $g(z, w)$ is any polynomial in the formal variables z and w ; $r, s \in \mathbf{N}$, $a_l \neq 0$ for $l = 1, \dots, r$. We can define two embeddings of such rational functions into the space $W[[z, z^{-1}, w, w^{-1}]]$ of formal Laurent series in z and w . The first of these is designated as ι_{zw} and obtained by expanding the product factors in (2.27) in non-negative integral powers of the variable w . The second, denoted by ι_{wz} , is obtained by pulling out the z factors in the denominator of (2.27) instead and expanding in non-negative powers of z . We note that even for the same rational function $f(z, w)$ the formal power series $\iota_{zw}f$ and $\iota_{wz}f$ will differ as elements of $W[[z, z^{-1}, w, w^{-1}]]$ unless f is a Laurent polynomial.

We have the following important theorem due to [21].

Theorem 1

1. (Locality \equiv rationality of products + commutativity)

For $\chi, \psi, \varphi, \xi \in \mathcal{F}$, the formal series $\langle \chi | \mathcal{V}(\psi, z) \mathcal{V}(\varphi, w) \xi \rangle$ which involves only finitely many negative powers of w and only finitely many positive powers of z , lies in the image of the map ι_{zw} :

$$\langle \chi | \mathcal{V}(\psi, z) \mathcal{V}(\varphi, w) \xi \rangle = \iota_{zw} f(z, w), \quad (2.28)$$

where the (uniquely determined) element $f \in \mathbf{C}[z, w]_S$ is of the form

$$f(z, w) = \frac{g(z, w)}{z^r w^s (z - w)^t} \quad (2.29)$$

for some polynomial $g(z, w) \in \mathbf{C}[z, w]$ and $r, s, t \in \mathbf{N}$. We also have

$$\langle \chi | \mathcal{V}(\varphi, w) \mathcal{V}(\psi, z) \xi \rangle = \iota_{wz} f(z, w), \quad (2.30)$$

i.e. $\mathcal{V}(\psi, z) \mathcal{V}(\varphi, w)$ agrees with $\mathcal{V}(\varphi, w) \mathcal{V}(\psi, z)$ as operator-valued rational functions.

2. (Duality \equiv rationality of iterates + associativity)

For $\chi, \psi, \varphi, \xi \in \mathcal{F}$, the formal series $\langle \chi | \mathcal{V}(\mathcal{V}(\psi, y) \varphi, w) \xi \rangle$ which involves only finitely many negative powers of y and only finitely many positive powers of w , lies in the image of the map ι_{wy} :

$$\langle \chi | \mathcal{V}(\mathcal{V}(\psi, y) \varphi, w) \xi \rangle = \iota_{wy} f(y + w, w), \quad (2.31)$$

with the same f as above, and

$$\langle \chi | \mathcal{V}(\psi, y + w) \mathcal{V}(\varphi, w) \xi \rangle = \iota_{yw} f(y + w, w), \quad (2.32)$$

i.e. $\mathcal{V}(\psi, z) \mathcal{V}(\varphi, w)$ and $\mathcal{V}(\mathcal{V}(\psi, z - w) \varphi, w)$ agree as operator-valued rational functions, where the right hand expression is to be expanded as a Laurent series in $z - w$.

For a proof see [21] or [19].

The first part of the theorem in particular states that these matrix elements may be viewed as meromorphic functions of the formal variables. In other words, for (2.28) and (2.30) there exist meromorphic functions of z and w which upon expansion in z or w agree with the formal power series on the left hand side in their respective domains of convergence. Thus vertex algebras can be seen as a rigorous formulation of meromorphic conformal field theories. Note that the second part of the theorem should be interpreted as crossing symmetry of the four-point correlation function on the Riemann sphere. It establishes a precise formulation of an operator product expansion in two-dimensional conformal field theory (see e.g. [42], [4], [29], [31]) in the sense that $\mathcal{V}(\psi, z) \mathcal{V}(\varphi, w)$ agrees with $\sum_{n \in \mathbf{Z}} (z - w)^{-n-1} \mathcal{V}(\psi_n \varphi, w)$ as an operator-valued rational function. The theorem then also ensures that this operator product expansion involves only finitely many singular (at “ $z = w$ ”) terms.

For establishing the Jacobi identity in concrete models the following theorem is particularly useful. (For a proof see [19])

Theorem 2

1. *The Jacobi identity follows from locality and duality.*
2. *In the definition of a vertex algebra the Jacobi identity may be replaced by the principle of locality, Eq. (2.12) and Eq. (2.13).*

If we regard our formal variables as *complex* variables, then the formal expansions of rational functions that we have been discussing converge in suitable domains. The matrix elements in Eq. (2.28) and Eq. (2.30) converge to a common rational function in the disjoint domains $|z| > |w| > 0$ and $|w| > |z| > 0$, respectively. The matrix elements in Eq. (2.31) and Eq. (2.32) for $y = z - w$ converge to a common rational function in the domains $|w| > |z - w| > 0$ and $|z| > |w| > 0$, respectively, and in the common domain $|z| > |w| > |z - w| > 0$ these two series converge to the common function.

We will now briefly discuss the issue of invariant bilinear form and adjoint vertex operator for a vertex algebra. Define the **restricted dual** of \mathcal{F} ,

$$\mathcal{F}' \equiv \bigoplus_{n \in \mathbf{Z}} (\mathcal{F}^n)^*, \quad (2.33)$$

the direct sum of the dual spaces of the homogeneous subspaces \mathcal{F}^n , i.e. the space of linear functionals on the vertex algebra \mathcal{F} vanishing on all but finitely many \mathcal{F}^n . Suppose we have a grading-preserving linear isomorphism $\mathcal{R} : \mathcal{F} \rightarrow \mathcal{F}'$, $\psi \mapsto \mathcal{R}(\psi) \equiv \psi^*$, satisfying $\mathcal{R} \circ \psi_n = \psi_n^* \circ \mathcal{R}$, then this amounts to choosing a nondegenerate bilinear form $(\cdot | \cdot)$ on \mathcal{F} as $(\chi | \varphi) := \langle \mathcal{R}(\chi) | \varphi \rangle$ where $\langle \cdot | \cdot \rangle$ denotes the natural pairing between \mathcal{F} and \mathcal{F}' . We may define the **adjoint vertex operator** w.r.t. the bilinear form by stipulating $(\mathcal{V}(\psi, z)\chi | \varphi) = (\chi | \mathcal{V}^\sharp(\psi, z)\varphi)$ and putting (cf. [19])

$$\begin{aligned} \mathcal{V}^\sharp(\psi, z) &= \sum_{n \in \mathbf{Z}} \psi_n^\sharp z^{-n-1} \\ &:= \mathcal{V}(e^{zL_1}(-z^{-2})^{L_0}\psi, z^{-1}) \in (\text{End } \mathcal{F})[[z, z^{-1}]]. \end{aligned} \quad (2.34)$$

We observe that the expression on the right hand side in general is not summable unless $(L_1)^n \psi = 0$ for n large enough, which is assured if the spectrum of L_0 is bounded from below. For the string moving on a Minkowski torus the spectrum of L_0 is unbounded. But for our purposes it is sufficient to know the adjoint vertex operator associated with quasiprimary states, and for these states the above definition indeed makes sense:

$$\mathcal{V}^\sharp(\psi, z) = (-1)^{h_\psi} z^{-2h_\psi} \mathcal{V}(\psi, z^{-1}) \quad \text{for } \psi \text{ quasiprimary}, \quad (2.35)$$

or, in terms of the mode expansion (2.11),

$$\psi_n^\sharp = (-1)^{h_\psi} \psi_{-n+2h_\psi-2}. \quad (2.36)$$

The Virasoro generators satisfy the relation $L_n^\sharp = L_{-n}$ (note the special mode expansion (2.6) for the Virasoro generators!) or, in terms of the “stress-energy tensor”, $\mathcal{T}^\sharp(z) = \frac{1}{z^4} \mathcal{T}(z^{-1})$. From this it is easy to see that the homogeneous subspaces \mathcal{F}^n , $n \in \mathbf{Z}$, are orthogonal to each other with respect to this bilinear form,

$$(\mathcal{F}^m | \mathcal{F}^n) = 0 \quad \text{if } m \neq n. \quad (2.37)$$

One can also show that the bilinear form is symmetric and that the conjugation operation \sharp satisfies $\mathcal{V}^{\sharp\sharp} = \mathcal{V}$ (for proofs see [19]).

We shall provide a certain subspace of the Fock space \mathcal{F} with the structure of a Lie algebra (cf. [5], [21]). We define a bilinear product on \mathcal{F} by

$$[\psi, \varphi] := \text{Res}_z [\mathcal{V}(\psi, z)\varphi] \equiv \psi_0 \varphi, \quad (2.38)$$

which is antisymmetric on the quotient space $\mathcal{F}/L_{-1}\mathcal{F}$ due to the skew symmetry property (2.16). By considering the mode expansion of the Jacobi identity we get $\psi_0(\varphi_0\xi) - \varphi_0(\psi_0\xi) = (\psi_0\varphi)_0\xi$. But this equation translates precisely into the classical Jacobi identity for Lie algebras,

$$[[\psi, \varphi], \xi] + [[\varphi, \xi], \psi] + [[\xi, \psi], \varphi] = 0, \quad (2.39)$$

on $\mathcal{F}/L_{-1}\mathcal{F}$. Hence, (2.38) defines a Lie bracket on this quotient space. At the level of vertex operators, dividing out the subspace $L_{-1}\mathcal{F}$ reflects the fact that the zero mode ψ_0 of a vertex operator $\mathcal{V}(\psi, z)$ remains

unchanged when a total derivative is added to $\mathcal{V}(\psi, z)$; for if $\psi = L_{-1}\varphi \in L_{-1}\mathcal{F}$ for some $\varphi \in \mathcal{F}$, then $\psi_0 = \text{Res}_z \left[\frac{d}{dz} \mathcal{V}(\varphi, z) \right] = 0$ by (2.8) and (2.1). If one prefers a representation of a Lie algebra in terms of linear operators we point out that the operator $\psi_0 = \text{Res}_z [\mathcal{V}(\psi, z)]$ is just the adjoint action of ψ on $\mathcal{F}/L_{-1}\mathcal{F}$, viz.

$$\text{ad}_\psi(\varphi) = [\psi, \varphi] = \psi_0(\varphi). \quad (2.40)$$

In other words the Lie algebra of zero mode operators, $\{\psi_0 \mid \psi \in \mathcal{F}\}$, is the adjoint representation of $\mathcal{F}/L_{-1}\mathcal{F}$. Consequently, the above Jacobi identity establishes the homomorphism property of the map $\mathcal{F}/L_{-1}\mathcal{F} \rightarrow \text{End } \mathcal{F}$, $\psi \mapsto \psi_0$:

$$\text{ad}_{[\psi, \varphi]} = ([\psi, \varphi])_0 = [\psi_0, \varphi_0] = [\text{ad}_\psi, \text{ad}_\varphi]. \quad (2.41)$$

The Lie algebra $\mathcal{F}/L_{-1}\mathcal{F}$ will be too large for our purposes. In string theory, a distinguished role is played by the primary states of weight $h = 1$, which we shall call **physical states** from now on. In fact, we learn from Eq. (2.21) that for a physical state ψ the corresponding zero mode operator ψ_0 commutes with the Virasoro algebra thereby preserving all subspaces \mathcal{P}^n of primary states of weight n . In particular, it maps physical states into physical states, i.e. $[\mathcal{P}^1, \mathcal{P}^1] \subset \mathcal{P}^1 \bmod L_{-1}\mathcal{P}^0$. Hence we will be mainly interested in the **Lie algebra of physical states**,

$$\mathbf{g}_{\mathcal{F}} := \mathcal{P}^1/L_{-1}\mathcal{P}^0, \quad (2.42)$$

where we used the fact that

$$L_{-1}\mathcal{F}^0 \cap \mathcal{P}^1 = L_{-1}\mathcal{P}^0 \quad (2.43)$$

in any vertex algebra (see [24] for the proof).

For the benefit of readers not familiar with the abstract formalism of vertex algebras we pause to briefly explain the connection between the formula (2.38) and the definition used in [32]. There the commutator of two integrated vertex operators is defined by means of the following contour integrals

$$[\psi_0, \varphi_0] := \oint_{|z| > |w|} \frac{dz}{2\pi i} \mathcal{V}(\psi, z) \oint_0 \frac{dw}{2\pi i} \mathcal{V}(\varphi, w) - \oint_{|w| > |z|} \frac{dw}{2\pi i} \mathcal{V}(\varphi, w) \oint_0 \frac{dz}{2\pi i} \mathcal{V}(\psi, z). \quad (2.44)$$

The arrangement of contours is important here because the operator products must be expanded in their respective domains of convergence for the above expressions to make sense; the two expansions correspond precisely to the operations ι_{zw} and ι_{wz} in the calculus of formal power series introduced above. To evaluate (2.44) we recall the locality property of vertex operators, i.e. $\mathcal{V}(\psi, z)\mathcal{V}(\varphi, w)$ and $\mathcal{V}(\varphi, w)\mathcal{V}(\psi, z)$ are the same as operator-valued rational functions. Hence we may deform the contour such that the two contributions can be combined into a single one to obtain [32]

$$[\psi_0, \varphi_0] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} \mathcal{V}(\psi, z) \mathcal{V}(\varphi, w), \quad (2.45)$$

where the integral over z is to be performed with a small contour encircling the point w . By duality (i.e. Part 2 of Theorem 1), this expression can be rewritten as

$$\begin{aligned} [\psi_0, \varphi_0] &= \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} \mathcal{V}(\mathcal{V}(\psi, z-w)\varphi, w) \\ &= \oint_0 \frac{dw}{2\pi i} \oint_0 \frac{dz}{2\pi i} \mathcal{V}(\mathcal{V}(\psi, z)\varphi, w). \end{aligned} \quad (2.46)$$

Hence the resulting vertex operator coincides with the operator $(\psi_0\varphi)_0$ corresponding to the state (2.38), and therefore the definition (2.38) and the commutator prescription are entirely equivalent.

The problem of finding an invariant bilinear form for the Lie algebra of physical states is resolved quite elegantly; for it turns out that the bilinear form $(-|-)$ defined above gives rise to an invariant bilinear form on $\mathbf{g}_{\mathcal{F}}$. We first have to convince ourselves that the form on \mathcal{F} projects down to a well-defined form on $\mathbf{g}_{\mathcal{F}}$. Indeed, since $L_{-1}^\dagger = L_1$ we have $(L_{-1}\chi|\varphi) = 0$ for any quasiprimary state φ , i.e. the space $L_{-1}\mathcal{F}$ is orthogonal to all quasiprimary states and thus deserves to be called null space. In particular, $L_{-1}\mathcal{P}^0$ consists of **null physical**

states, physical states orthogonal to all physical states including themselves. Hence the Lie algebra $\mathfrak{g}_{\mathcal{F}}$ is obtained from \mathcal{P}^1 by dividing out (unwanted) null physical states. Recall that when defining the Lie algebra $\mathcal{F}/L_{-1}\mathcal{F}$ we had to divide out the space $L_{-1}\mathcal{F}$ for mathematical reasons. But with the bilinear form at hand we are now led to a physical interpretation of that maneuver.

It is well known that there are additional null physical states in \mathcal{P}^1 if and only if the central charge takes the critical value $c = 26$, namely the space $(L_{-2} + \frac{3}{2}L_{-1}^2)\mathcal{P}^{-1}$ (see [35] for the calculations). The existence of these additional null physical states is used in the proof of the no-ghost theorem [34] which we shall refer to in the last chapter when we exploit the DDF construction.

As already mentioned, the adjoint vertex operator in Eq. (2.34) is summable for any quasiprimary state irrespective of the spectrum of L_0 . Together with Eq. (2.36), this implies that the form always satisfies $(\chi|\psi_n\varphi) = -(\psi_{-n}\chi|\varphi)$ for physical states χ, ψ, φ . Specializing to $n = 0$ we indeed recover the invariance property:

$$([\chi, \psi]|\varphi) = (\chi|[\psi, \varphi]) \quad \forall \chi, \psi, \varphi \in \mathfrak{g}_{\mathcal{F}}. \quad (2.47)$$

If we put $n = 1$ and $\chi = \mathbf{1}$, then we obtain

$$(\psi|\varphi) = (\mathbf{1}|\psi_1\varphi) \quad \forall \psi, \varphi \in \mathfrak{g}_{\mathcal{F}}, \quad (2.48)$$

so that we can think of the projection of $-\psi_1\varphi$ on the vacuum as the bilinear form.

2.2 Toroidal compactification of the bosonic string

It is by no means obvious that nontrivial examples of vertex algebras exist. However, an important class of vertex algebras is provided by the following result, which was announced in [5] and was proved in [21].

Theorem 3

Associated with each nondegenerate even lattice Λ there is a vertex algebra.

In fact, the above examples of vertex algebras gave rise to the very notion and the abstract definition of vertex algebras. As we shall see below, the physics described by these vertex algebras is the chiral sector of a first quantized closed bosonic string moving on a spacetime torus. This section will be concerned with the explicit construction of the vertex algebra stated above and the discussion of the Lie algebra of physical states with the invariant and covariant bilinear forms. For further details, the reader may also wish to consult the articles [32], [30] and [33] or the comprehensive review [40].

Let Λ be an even lattice of rank $d < \infty$ with a symmetric nondegenerate \mathbf{Z} -valued \mathbf{Z} -bilinear form \cdot, \cdot and corresponding metric tensor $\eta^{\mu\nu}$, $1 \leq \mu, \nu \leq d$ (Λ even means that $\mathbf{r}^2 \in 2\mathbf{Z}$ for all $\mathbf{r} \in \Lambda$). The vertex algebra which we shall construct can be thought of as a chiral sector of a first quantized closed bosonic string theory with d spacetime dimensions compactified on a torus. Thus Λ represents the allowed momentum vectors of the theory.

Introduce **oscillators** α_m^μ , $m \in \mathbf{Z}$, $1 \leq \mu \leq d$, satisfying the commutation relations of a d -fold **Heisenberg algebra**,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad (2.49)$$

and **zero mode states** $\Psi_{\mathbf{r}}$, $\mathbf{r} \in \Lambda$. We want the latter to carry momentum \mathbf{r} and to be annihilated by the positive oscillator modes, i.e.

$$\alpha_m^\mu \Psi_{\mathbf{r}} = 0 \quad \text{if } m > 0, \quad (2.50)$$

$$p^\mu \Psi_{\mathbf{r}} = r^\mu \Psi_{\mathbf{r}}, \quad (2.51)$$

where $p^\mu \equiv \alpha_0^\mu$ denotes the center of mass momentum operator for the string and r^μ are the components of $\mathbf{r} \in \Lambda$. While the operators α_m^μ for $m > 0$ by definition act as annihilation operators, the operators α_m^μ for $m < 0$ will be called creation operators, since they generate an irreducible Heisenberg module $\mathcal{F}^{(\mathbf{r})}$ with highest weight $\mathbf{r} \in \Lambda$ from any ground state $\Psi_{\mathbf{r}}$.

We take $\Lambda_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Z}} \Lambda$ to be the real vector space in which Λ is embedded, and for notational convenience we define

$$\mathbf{r}(m) := \sum_{\mu=1}^d r_\mu \alpha_m^\mu \equiv \mathbf{r} \cdot \boldsymbol{\alpha}_m \quad (2.52)$$

for $\mathbf{r} \in \Lambda_{\mathbf{R}}$, $m \in \mathbf{Z}$, such that

$$[\mathbf{r}(m), \mathbf{s}(n)] = m(\mathbf{r} \cdot \mathbf{s}) \delta_{m+n,0}, \quad (2.53)$$

with the \mathbf{Z} -bilinear form on Λ to be extended to an \mathbf{R} -bilinear form on $\Lambda_{\mathbf{R}}$. We denote the d -fold Heisenberg algebra spanned by the oscillators by

$$\hat{\mathbf{h}} := \{\mathbf{r}(m) \mid \mathbf{r} \in \Lambda_{\mathbf{R}}, m \in \mathbf{Z}\} \oplus \mathbf{R} \cdot 1, \quad (2.54)$$

and for the vector space of finite products of creation operators (\equiv algebra of polynomials on the negative oscillator modes) we write

$$S(\hat{\mathbf{h}}^-) := \bigoplus_{N \in \mathbf{N}} \left\{ \prod_{i=1}^N \mathbf{r}_i(-m_i) \mid \mathbf{r}_i \in \Lambda_{\mathbf{R}}, m_i > 0 \text{ for } 1 \leq i \leq N \right\}, \quad (2.55)$$

where “ S ” stands for “symmetric” because of the fact that the creation operators commute with each other. Hence the Heisenberg module built on some ground state $\Psi_{\mathbf{r}}$ is given by $\mathcal{F}^{(\mathbf{r})} = S(\hat{\mathbf{h}}^-) \Psi_{\mathbf{r}}$.

If we formally introduce center of mass position operators q^μ , $1 \leq \mu \leq d$, commuting with α_m^μ for $m \neq 0$ and satisfying

$$[q^\nu, p^\mu] = i\eta^{\mu\nu}, \quad (2.56)$$

then we find that

$$e^{i\mathbf{r} \cdot \mathbf{q}} \Psi_{\mathbf{s}} = \Psi_{\mathbf{r}+\mathbf{s}}, \quad (2.57)$$

i.e. the zero mode states can be generated from the vacuum Ψ_0 :

$$\Psi_{\mathbf{r}} = e^{i\mathbf{r} \cdot \mathbf{q}} \Psi_0. \quad (2.58)$$

Thus the operators $e^{i\mathbf{r} \cdot \mathbf{q}}$, $\mathbf{r} \in \Lambda$, may be identified with the zero mode states and form an Abelian group which is called the **group algebra** of the lattice Λ and is denoted by $\mathbf{R}[\Lambda]$. Collecting all the Heisenberg modules $\mathcal{F}^{(\mathbf{r})}$ one might expect the full Fock space \mathcal{F} of the vertex algebra to be $S(\hat{\mathbf{h}}^-) \otimes \mathbf{R}[\Lambda]$. However, it is well-known that we need to replace the group algebra $\mathbf{R}[\Lambda]$ by something more delicate in order to adjust the signs in the Jacobi identity for the vertex algebra. We will multiply $e^{i\mathbf{r} \cdot \mathbf{q}}$ by a so-called **cocycle factor** $c_{\mathbf{r}}$ which is a function of momentum \mathbf{p} . This means that it commutes with all oscillators α_m^μ and satisfies the eigenvalue equations

$$c_{\mathbf{r}} \Psi_{\mathbf{s}} = \epsilon(\mathbf{r}, \mathbf{s}) \Psi_{\mathbf{s}}. \quad (2.59)$$

This can be implemented by imposing the conditions

$$\epsilon(\mathbf{r}, \mathbf{s}) \epsilon(\mathbf{r} + \mathbf{s}, \mathbf{t}) = \epsilon(\mathbf{r}, \mathbf{s} + \mathbf{t}) \epsilon(\mathbf{s}, \mathbf{t}), \quad (2.60)$$

$$\epsilon(\mathbf{r}, \mathbf{s}) = (-1)^{\mathbf{r} \cdot \mathbf{s}} \epsilon(\mathbf{s}, \mathbf{r}), \quad (2.61)$$

$$\epsilon(\mathbf{r}, -\mathbf{r}) = (-1)^{\frac{1}{2} \mathbf{r}^2}, \quad (2.62)$$

$$\epsilon(\mathbf{0}, \mathbf{0}) = 1. \quad (2.63)$$

Note that the cocycle condition (2.60) implies $\epsilon(\mathbf{0}, \mathbf{0}) = \epsilon(\mathbf{0}, \mathbf{r}) = \epsilon(\mathbf{r}, \mathbf{0}) \forall \mathbf{r}$. Without loss of generality we can assume that the function ϵ is bimultiplicative, i.e. $\epsilon(\mathbf{r} + \mathbf{s}, \mathbf{t}) = \epsilon(\mathbf{r}, \mathbf{t}) \epsilon(\mathbf{s}, \mathbf{t})$ and $\epsilon(\mathbf{r}, \mathbf{s} + \mathbf{t}) = \epsilon(\mathbf{r}, \mathbf{s}) \epsilon(\mathbf{r}, \mathbf{t}) \forall \mathbf{r}, \mathbf{s}, \mathbf{t}$. Together with (2.62) and the normalization condition (2.63), this then implies that $\epsilon(m\mathbf{r}, n\mathbf{r}) = [\epsilon(\mathbf{r}, \mathbf{r})]^{mn} = (-1)^{\frac{1}{2} mn \mathbf{r}^2} \forall \mathbf{r}$, $m, n \in \mathbf{Z}$. We will take the **twisted group algebra** $\mathbf{R}\{\Lambda\}$ consisting of the operators

$$e_{\mathbf{r}} := e^{i\mathbf{r} \cdot \mathbf{q}} c_{\mathbf{r}}, \quad (2.64)$$

for $\mathbf{r} \in \Lambda$, instead of $\mathbf{R}[\Lambda]$. This means just that we are working with a certain section in the double cover $\hat{\Lambda}$ of the lattice Λ . Due to the identification of the operators $e_{\mathbf{r}}$ with zero mode states it is suggestive to adopt the ket notation $e_{\mathbf{r}} \equiv |\mathbf{r}\rangle$ which we will use where appropriate.

In summary, the Fock space associated with the lattice Λ is defined to be

$$\mathcal{F} := S(\hat{\mathbf{h}}^-) \otimes \mathbf{R}\{\Lambda\}. \quad (2.65)$$

Note that the oscillators $\mathbf{r}(m)$, $m \neq 0$, act only on the first tensor factor, namely, creation operators as multiplication operators and annihilation operators via the adjoint representation, i.e. by (2.53). The zero mode operators α_0^μ , however, are only sensible for the twisted group algebra, viz.

$$\mathbf{r}(0)|\mathbf{s}\rangle = (\mathbf{r} \cdot \alpha_0)|\mathbf{s}\rangle = (\mathbf{r} \cdot \mathbf{s})|\mathbf{s}\rangle \quad \forall \mathbf{r} \in \Lambda_{\mathbf{R}}, \mathbf{s} \in \Lambda, \quad (2.66)$$

while the action of $e_{\mathbf{r}}$ on $\mathbf{R}\{\Lambda\}$ is given by

$$e_{\mathbf{r}}e_{\mathbf{s}} = \epsilon(\mathbf{r}, \mathbf{s})e_{\mathbf{r}+\mathbf{s}}. \quad (2.67)$$

We shall define next the **vertex operators** $\mathcal{V}(\psi, z)$ for $\psi \in \mathcal{F}$. We introduce the **Fubini–Veneziano coordinate field**,

$$X^\mu(z) := q^\mu - ip^\mu \ln z + i \sum_{m \in \mathbf{Z}} \frac{1}{m} \alpha_m^\mu z^{-m}, \quad (2.68)$$

which really only has a meaning when exponentiated, and the **Fubini–Veneziano momentum field**,

$$P^\mu(z) := i \frac{d}{dz} X^\mu(z) = \sum_{m \in \mathbf{Z}} \alpha_m^\mu z^{-m-1}. \quad (2.69)$$

For $\mathbf{r} \in \Lambda_{\mathbf{R}}$ we define the formal sum

$$\mathbf{r}(z) \equiv \mathbf{r} \cdot \mathbf{P}(z) = \sum_{m \in \mathbf{Z}} \mathbf{r}(m) z^{-m-1}, \quad (2.70)$$

which is an element of $\hat{\mathbf{h}}[[z, z^{-1}]]$ and may be regarded as a generating function for the operators $\mathbf{r}(m)$, $m \in \mathbf{Z}$, or as a “current” in contrast to the “states” in \mathcal{F} . It is convenient to split $X^\mu(z)$ into three parts:

$$X^\mu(z) = X_{<}^\mu(z) + (q^\mu - ip^\mu \ln z) + X_{>}^\mu(z), \quad (2.71)$$

where

$$X_{<}^\mu(z) := -i \sum_{m>0} \frac{1}{m} \alpha_{-m}^\mu z^m, \quad X_{>}^\mu(z) := i \sum_{m>0} \frac{1}{m} \alpha_m^\mu z^{-m}. \quad (2.72)$$

We will employ the usual normal-ordering procedure, i.e. colons indicate that in the enclosed expressions, q^ν is written to the left of p^μ , as well as the creation operators are to be placed to the left of the annihilation operators:

$$\begin{aligned} :\mathbf{r}(m)\mathbf{s}(n): &= \begin{cases} \mathbf{r}(m)\mathbf{s}(n) & \text{if } m \leq n, \\ \mathbf{s}(n)\mathbf{r}(m) & \text{if } m > n, \end{cases} \\ :q^\nu p^\mu: &= :p^\mu q^\nu: = q^\nu p^\mu. \end{aligned} \quad (2.73)$$

For $|\mathbf{r}\rangle \in \mathbf{R}\{\Lambda\}$, the associated vertex operator then takes the familiar form

$$\begin{aligned} \mathcal{V}(|\mathbf{r}\rangle, z) &:= e^{i\mathbf{r} \cdot \mathbf{X}_{<}(z)} e_{\mathbf{r}} z^{\mathbf{r}(0)} e^{i\mathbf{r} \cdot \mathbf{X}_{>}(z)} \\ &= :e^{i\mathbf{r} \cdot \mathbf{X}(z)}: c_{\mathbf{r}}, \end{aligned} \quad (2.74)$$

Note that the cocycle factors of the vertex operators are hidden in the elements of the twisted group algebra, $\mathbf{R}\{\Lambda\}$.

Let $\psi = [\prod_{j=1}^N \mathbf{s}_j(-n_j)]|\mathbf{r}\rangle$ be a typical homogeneous element of \mathcal{F} and define

$$\begin{aligned} \mathcal{V}(\psi, z) &:= : \mathcal{V}(|\mathbf{r}\rangle, z) \prod_{j=1}^N \frac{1}{(n_j - 1)!} \left(\frac{d}{dz} \right)^{n_j-1} (\mathbf{s}_j \cdot \mathbf{P}(z)) : \\ &\equiv i : e^{i\mathbf{r} \cdot \mathbf{X}(z)} \prod_{j=1}^N \frac{1}{(n_j - 1)!} \left(\frac{d}{dz} \right)^{n_j} (\mathbf{s}_j \cdot \mathbf{X}(z)) : c_{\mathbf{r}}. \end{aligned} \quad (2.75)$$

Extending this definition by linearity we finally obtain a well-defined map

$$\begin{aligned}\mathcal{V} : \mathcal{F} &\rightarrow (\text{End } \mathcal{F})[[z, z^{-1}]], \\ \psi &\mapsto \mathcal{V}(\psi, z) = \sum_{n \in \mathbf{Z}} \psi_n z^{-n-1}.\end{aligned}\tag{2.76}$$

We choose the vacuum to be the zero mode state with no momentum and without any creation operators, i.e.

$$\mathbf{1} := |\mathbf{0}\rangle \equiv 1 \otimes e_0.\tag{2.77}$$

The state

$$\omega := \frac{1}{2} \sum_{\mu=1}^d \eta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu |\mathbf{0}\rangle \equiv \frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} |\mathbf{0}\rangle\tag{2.78}$$

provides a conformal vector of dimension d . Indeed, by (2.75) and (2.70), we have

$$\mathcal{T}(z) \equiv \mathcal{V}(\omega, z) = \frac{1}{2} \sum_{\mu=1}^d : \mathbf{P}(z) \cdot \mathbf{P}(z) :$$

so that

$$L_n \equiv \omega_{n+1} = \frac{1}{2} \sum_{m \in \mathbf{Z}} : \alpha_m \cdot \alpha_{n-m} : ,\tag{2.79}$$

in agreement with the well-known expression from string theory. Using the oscillator commutation relations one finds that the L_n 's obey (2.7) with central charge $c = d$ (see e.g. [35] for the calculation).

Finally, let $\psi = [\prod_{j=1}^N \mathbf{s}_j(-n_j)] |\mathbf{r}\rangle$ be a typical homogeneous element of \mathcal{F} . Then

$$L_0 \psi = \left(\frac{1}{2} \mathbf{r}^2 + \sum_{j=1}^N n_j \right) \psi\tag{2.80}$$

yields the desired grading of \mathcal{F} . We observe that if Λ is Lorentzian then \mathbf{r}^2 can be arbitrarily negative so that the spectrum of L_0 is unbounded from above as well as from below.

With the above definitions it is straightforward to verify the first four axioms for a vertex algebra, but to prove the Jacobi identity, into which most of the information about the vertex algebra is encoded, is much harder. A proof based on the old normal-ordering string techniques can be found in [21]. In Sect. 4.2 we will present an alternative much more elegant proof using overlap identities for the three-vertex.

We turn now to the analysis of the Lie algebra of physical states, \mathbf{g}_Λ . The simplest physical states are the **tachyonic states** $\mathbf{g}_\Lambda^{[0]} := \{|\mathbf{r}\rangle \mid \mathbf{r} \in \Lambda, \mathbf{r}^2 = 2\}$ and the **photonic states**: $\mathbf{g}_\Lambda^{[1]} := \{\mathbf{s}(-1)|\mathbf{r}\rangle \mid \mathbf{r} \cdot \mathbf{s} = 0, \mathbf{s} \in \Lambda_{\mathbf{R}}, \mathbf{r} \in \Lambda, \mathbf{r}^2 = 0\}$, where the superscript of \mathbf{g}_Λ counts the oscillator excitations. We want to stress again that the physical states in \mathbf{g}_Λ are only defined modulo $L_{-1}\mathcal{P}^0$, which means for example that $\mathbf{r}(-1)|\mathbf{r}\rangle = L_{-1}|\mathbf{r}\rangle \equiv 0$ in \mathbf{g}_Λ for a vector $\mathbf{r} \in \Lambda$ with $\mathbf{r}^2 = 0$.

The commutators between the first nontrivial physical states become

$$[\rho(-1)|\mathbf{0}\rangle, \sigma(-1)|\mathbf{0}\rangle] = 0,\tag{2.81}$$

$$[\rho(-1)|\mathbf{0}\rangle, |\mathbf{r}\rangle] = (\rho \cdot \mathbf{r})|\mathbf{r}\rangle,\tag{2.82}$$

$$[|\mathbf{r}\rangle, |\mathbf{s}\rangle] = \begin{cases} 0 & \text{if } \mathbf{r} \cdot \mathbf{s} \geq 0, \\ \epsilon(\mathbf{r}, \mathbf{s})|\mathbf{r} + \mathbf{s}\rangle & \text{if } \mathbf{r} \cdot \mathbf{s} = -1, \\ -\mathbf{r}(-1)|\mathbf{0}\rangle & \text{if } \mathbf{r} \cdot \mathbf{s} = -2, \\ \dots & \text{if } \mathbf{r} \cdot \mathbf{s} \leq -3, \end{cases}\tag{2.83}$$

for $\rho, \sigma \in \Lambda_{\mathbf{R}}$ and $\mathbf{r}, \mathbf{s} \in \Lambda, \mathbf{r}^2 = \mathbf{s}^2 = 2$. Note that for an Euclidean lattice the Schwarz inequality would imply $|\mathbf{r} \cdot \mathbf{s}| \leq 2$ leading to a finite-dimensional Lie algebra of physical states; but here we are interested in the much more complicated case of a Lorentzian lattice.

We have seen that a special role is played by the norm 2 vectors of Λ which we call **real roots** of the lattice. The **reflection** $\boxminus_{\mathbf{r}}$ associated with a real root \mathbf{r} is defined as $\boxminus_{\mathbf{r}}(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{r})\mathbf{r}$ for $\mathbf{x} \in \Lambda_{\mathbf{R}}$. It is easy to see that a reflection in a real root is an automorphism of the lattice. The hyperplanes perpendicular to these real

roots divide the vector space $\Lambda_{\mathbf{R}}$ into regions called **Weyl chambers**. The reflections in the real roots of Λ generate a group called the **Weyl group** \mathcal{W} of Λ , which acts simply transitively on the Weyl chambers of Λ . This means that if we fix one Weyl chamber \mathcal{C} once and for all, then any real root from the interior of another Weyl chamber can be transported via Weyl reflection to a unique real root in \mathcal{C} . The real roots \mathbf{r}_i that are perpendicular to the faces of \mathcal{C} and have inner product at most 0 with the elements of \mathcal{C} are called the **simple roots** of \mathcal{C} . The **(Coxeter-)Dynkin diagram** \mathcal{G} of \mathcal{C} is the set of simple roots of \mathcal{C} , drawn as a graph with one vertex for each simple root of \mathcal{C} and two vertices corresponding to the distinct roots $\mathbf{r}_i, \mathbf{r}_j$ are joined by $-\mathbf{r}_i \cdot \mathbf{r}_j$ lines.

Returning to the vertex algebra associated with the even Lorentzian lattice Λ , it is clear that, for any simple root \mathbf{r}_i , the elements $|\mathbf{r}_i\rangle$, $|- \mathbf{r}_i\rangle$, and $\mathbf{r}_i(-1)|\mathbf{0}\rangle$ describe physical states, i.e. they lie in \mathcal{P}^1 . Define generators for a Lie algebra $\mathfrak{g}(A)$ by

$$\begin{aligned} e_i &\mapsto |\mathbf{r}_i\rangle, \\ f_i &\mapsto -|- \mathbf{r}_i\rangle, \\ h_i &\mapsto \mathbf{r}_i(-1)|\mathbf{0}\rangle. \end{aligned} \tag{2.84}$$

Then, by (2.81) – (2.83), we find the following relations to hold:

$$[h_i, h_j] = 0, \tag{2.85}$$

$$[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \tag{2.86}$$

$$[e_i, f_j] = \delta_{ij}h_i, \tag{2.87}$$

where we defined the **Cartan matrix** $A = (a_{ij})$ associated with \mathcal{C} by $a_{ij} := \mathbf{r}_i \cdot \mathbf{r}_j$. The elements h_i obviously form a basis for an abelian subalgebra of $\mathfrak{g}(A)$ called the **Cartan subalgebra** $\mathfrak{h}(A)$. In technical terms, from the above commutators we learn that the elements $\{e_i, f_i, h_i | i\}$ generate the so-called free Lie algebra associated with A . But even more is true; for we can show that the **Serre relations**

$$(\text{ad } e_i)^{1-a_{ij}}e_j = 0, \quad (\text{ad } f_i)^{1-a_{ij}}f_j = 0, \tag{2.88}$$

are also fulfilled for all $i \neq j$. They follow from the physical state condition (2.18) by combining Eq. (2.80) with the natural Λ -gradation which the Lie algebra of physical states inherits from \mathcal{F} ,

$$\mathfrak{g}_{\Lambda}^{(\mathbf{x})} := \mathfrak{g}_{\Lambda} \cap S(\hat{\mathfrak{h}}^-)|\mathbf{x}\rangle, \tag{2.89}$$

for $\mathbf{x} \in \Lambda$. Of course, some of the subspaces $\mathfrak{g}_{\Lambda}^{(\mathbf{x})}$ may be empty, e.g. for $\mathbf{x}^2 > 2$; but if $\mathfrak{g}_{\Lambda}^{(\mathbf{x})}$ is nonempty we shall refer to $\mathbf{x} \in \Lambda$ as a **root** of \mathfrak{g}_{Λ} with **root space** $\mathfrak{g}_{\Lambda}^{(\mathbf{x})}$ and **multiplicity** $\dim \mathfrak{g}_{\Lambda}^{(\mathbf{x})}$. Hence the number of linearly independent polarization vectors for a physical state with certain momentum \mathbf{x} accounts for the multiplicity of \mathbf{x} as a root for the Lie algebra of physical states. Having established the Serre relations, the Gabber–Kac theorem [37, Theorem 9.11] tells us that the Lie algebra $\mathfrak{g}(A)$ generated by the elements $\{e_i, f_i, h_i | i\}$ is just the **Kac Moody algebra** associated with the Cartan matrix A . Namely, the latter is defined as the above free Lie algebra divided by the maximal ideal intersecting $\mathfrak{h}(A)$ trivially, and the theorem states that this maximal ideal is generated by the elements $\{(\text{ad } e_i)^{1-a_{ij}}e_j, (\text{ad } f_i)^{1-a_{ij}}f_j | i \neq j\}$.

We emphasize the remarkable fact that the physical state condition $L_0\psi = \psi$ accounts for all Serre relations whereas these are usually very difficult to deal with in the theory of Kac Moody algebras; or, in string theory language, the absence of particles with squared mass below the tachyon reflects the validity of the Serre relations for the Lie algebra $\mathfrak{g}(A)$.

To summarize (cf. [5]): The physical states $\{|\mathbf{r}_i\rangle, |- \mathbf{r}_i\rangle, \mathbf{r}_i(-1)|\mathbf{0}\rangle | i\}$ generate via multiple commutators the Kac Moody algebra $\mathfrak{g}(A)$ associated with the Cartan matrix $A = (\mathbf{r}_i \cdot \mathbf{r}_j)$ which is a subalgebra of the Lie algebra of physical states, \mathfrak{g}_{Λ} . Only in the Euclidean case these two Lie algebras coincide. In general, we have a *proper* inclusion

$$\mathfrak{g}(A) \hookrightarrow \mathfrak{g}_{\Lambda}, \tag{2.90}$$

and the characterization of the elements of \mathfrak{g}_{Λ} not contained in the Lie algebra $\mathfrak{g}(A)$ is the key problem for the vertex operator construction of hyperbolic Kac Moody algebras. In the following we shall refer to such physical states that cannot be obtained as linear combinations of multiple commutators of the generators of $\mathfrak{g}(A)$ as **missing** or **decoupled (physical) states**. For it will turn out that they are characterized by the property that they decouple from the S -matrix for $\mathfrak{g}(A)$, i.e. putting a missing state on one leg of the N -string vertex and saturating the other ones with elements of $\mathfrak{g}(A)$ will give zero for the matrix element.

The special feature of (2.90) is that the root system of the Kac Moody algebra $\mathfrak{g}(A)$ is well understood though its root multiplicities are not completely known for a single example; whereas the root system of \mathfrak{g}_Λ is a priori not related to that of a Kac Moody algebra although the root multiplicities are always known. Thus a complete understanding of (2.90) requires a “mechanism” which tells us how $\mathfrak{g}(A)$ has to be filled up with physical states to reach the complete Lie algebra of physical states.

Finally we turn to the construction of a nondegenerate bilinear form $(-|-)$ on \mathfrak{g}_Λ satisfying the condition $(\mathcal{V}(\psi, z)\chi|\varphi) = (\chi|\mathcal{V}^\sharp(\psi, z)\varphi)$ with the adjoint vertex operator as defined in (2.34). In fact, this “invariance” condition is strong enough to determine the bilinear form uniquely up to a normalization. First, we deduce that

$$\mathbf{r}(-m)^\sharp = -\mathbf{r}(m) \quad \text{or} \quad (\alpha_m^\mu)^\sharp = -\alpha_{-m}^\mu \quad (2.91)$$

for $\mathbf{r} \in \Lambda_\mathbf{R}$, $m \in \mathbf{Z}$. Thus the oscillator part of the bilinear form is uniquely fixed and it remains to calculate the zero modes. To do so, we evaluate $(\mathbf{r}(0)e_\mathbf{r}|e_\mathbf{s})$ and $(e_\mathbf{r}|\mathbf{s}(0)e_\mathbf{s})$ in both ways to find that $(e_\mathbf{r}|e_\mathbf{s})$ vanishes for $\mathbf{r}, \mathbf{s} \in \Lambda$ unless $\mathbf{r} = -\mathbf{s}$. On the other hand, an explicit computation using the vertex operators shows that $(e_\mathbf{r}|e_{-\mathbf{r}}) = (\mathbf{1}|\mathbf{1})$ for all $\mathbf{r} \in \Lambda$. Thus we have

$$(e_\mathbf{r}|e_\mathbf{s}) = \delta_{\mathbf{r}+\mathbf{s}, \mathbf{0}}(\mathbf{1}|\mathbf{1}), \quad (2.92)$$

which, together with (2.91), indeed uniquely fixes the bilinear form up to the normalization of $(\mathbf{1}|\mathbf{1})$. For reasons which will become clear in a moment, we shall choose the normalization

$$(\mathbf{1}|\mathbf{1}) := -1 \quad (2.93)$$

so that

$$\begin{aligned} (e_\mathbf{r}| - e_{-\mathbf{s}}) &= \delta_{\mathbf{r}, \mathbf{s}}, \\ \left(\mathbf{r}(-1)|\mathbf{0} \right| \mathbf{s}(-1)|\mathbf{0} \rangle) &= \mathbf{r} \cdot \mathbf{s} \end{aligned} \quad (2.94)$$

for $\mathbf{r}, \mathbf{s} \in \Lambda$. When we go over to the induced form on \mathfrak{g}_Λ these relations respectively give

$$\begin{aligned} (e_i|f_j) &= \delta_{ij}, \\ (h_i|h_j) &= a_{ij}, \end{aligned} \quad (2.95)$$

for the generators of the Kac Moody algebra $\mathfrak{g}(A)$. Together with Eq. (2.47) this shows that $(-|-)$ induces on $\mathfrak{g}(A)$ a **standard invariant bilinear form** [37]. This justifies our choice of normalization.

Whereas the bilinear form defined above yields a non-degenerate pairing of the states $|\mathbf{r}\rangle$ and $|\mathbf{-r}\rangle$, in physics we prefer a bilinear form which pairs $|\mathbf{r}\rangle$ with itself, because ultimately we are looking for a symmetric scalar product² which does not lead to physical states of negative norm. For this purpose we need the **Chevalley involution** θ which is given by

$$\begin{aligned} \theta(|\mathbf{r}\rangle) &:= |\mathbf{-r}\rangle, \\ \theta \circ \mathbf{r}(-m) \circ \theta^{-1} &:= -\mathbf{r}(-m) \quad \text{or} \quad \theta \circ \alpha_{-m}^\mu \circ \theta^{-1} := -\alpha_{-m}^\mu. \end{aligned} \quad (2.96)$$

Note that this definition of θ is consistent with the Chevalley involution on the Kac Moody algebra $\mathfrak{g}(A)$, viz.

$$\begin{aligned} \theta(e_i) &= -f_i, \\ \theta(f_i) &= -e_i, \\ \theta(h_i) &= -h_i. \end{aligned} \quad (2.97)$$

Since the Virasoro generators are bilinear in the oscillators it is clear that θ commutes with the L_n 's.

We introduce a **contravariant bilinear form** by defining

$$\langle \psi | \varphi \rangle := -(\theta(\psi)|\varphi) \quad (2.98)$$

for all $\psi, \varphi \in \mathcal{F}$. We immediately see that, with respect to $\langle -| - \rangle$, the zero mode states are orthonormal to each other and $\mathbf{r}(n)$ is the adjoint of $\mathbf{r}(-n)$,

$$\langle \mathbf{r} | \mathbf{s} \rangle = \delta_{\mathbf{r}, \mathbf{s}}, \quad (2.99)$$

$$\mathbf{r}(m)^\dagger = \mathbf{r}(-m) \quad \text{or} \quad (\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu. \quad (2.100)$$

²If we worked over the complex numbers, as one usually does in a Hilbert space, we would want an Hermitian scalar product.

Hence the contravariant bilinear form $\langle - | - \rangle$ is nothing but the familiar **string scalar product** to which the no-ghost theorem applies. In general we have

$$\mathcal{V}^\dagger(\psi, z) = \theta \circ \mathcal{V}^\sharp(\psi, z) \circ \theta^{-1} = \mathcal{V}^\sharp(\theta\psi, z), \quad (2.101)$$

due to the fact that θ is an automorphism of the vertex algebra. For example,

$$- [X^\mu(z)]^\dagger = [X^\mu(z)]^\sharp = X^\mu(z^{-1}), \quad (2.102)$$

$$- [P^\mu(z)dz]^\dagger = [P^\mu(z)dz]^\sharp = P^\mu(z^{-1})d(z^{-1}), \quad (2.103)$$

$$[\mathcal{T}(z)(dz)^2]^\dagger = [\mathcal{T}(z)(dz)^2]^\sharp = \mathcal{T}(z^{-1})[d(z^{-1})]^2, \quad (2.104)$$

so that $L_n^\dagger = L_n^\sharp = L_{-n}$ for all n . These identities suggest that, from the viewpoint of string theory, the \sharp -conjugation is more natural than the (usually preferred) \dagger -conjugation. Namely, whereas the latter comes from hermitian conjugation of the oscillators the former is nothing but the PT conjugation for it implements the transformation $z \rightarrow z^{-1}$ of the worldsheet coordinate $z = e^{t+ix}$.

3 General properties of multistring vertices

N -string vertices were first introduced as an efficient way of describing string scattering and computing amplitudes. In particular, their sewing properties ensured that factorization, and more specifically duality, were satisfied by the resulting string scattering amplitudes. However, their form and derivation were rather complicated and many of their properties unknown. One of the most important subsequent steps in the development of N -string vertices was the realization that they can be entirely characterized by a simple set of equations called overlap identities which they satisfy [48]. Whereas the computations required to determine the N -vertices by sewing used to be long and tedious in the early days of string theory (see e.g. [1]), the use of overlap identities simplifies the task enormously while at the same time leading to a beautiful geometrical picture. In addition they allowed for a simple derivation of all the properties of the N -string vertices. We can here provide only a brief summary; we shall, however, make an effort to present the pertinent results in a self-contained way and to explain the basic ideas as clearly as possible. In particular, we will explicitly show how to determine the vertices easily and efficiently by means of the overlap equations. In Sect. 5 we will see that, remarkably, the multistring formalism also furnishes the requisite tools for the analysis of hyperbolic Kac Moody algebras and their root spaces.

3.1 Basic definitions

It is well known that the amplitude for the scattering of N physical string states $\psi_1, \dots, \psi_N \in \mathcal{P}^1$ at tree level is obtained by evaluating the multistring vertex $V^{[N]}$ on these states and integrating the result with a suitable measure over the Koba Nielsen parameters³ z_1, \dots, z_N . For open and closed string tree level amplitudes, the integrals are to be carried out over real line and the whole complex plane (the Riemann sphere) for each z_i , respectively (see e.g. [35, 43] for further explanations). However, the calculational rules appropriate for our needs, namely the computation of Lie algebra multiple commutators, by necessity differ from the conventional ones. We are essentially dealing with the chiral half of a fully compactified closed string, and we accordingly replace the two-dimensional integrals by one-dimensional contour integrals. The relevant integrand can be regarded as the “holomorphic square root” of the corresponding formula for the closed string. We have

$$\begin{aligned} & W(\psi_1, \dots, \psi_N) \\ &= \oint \frac{dz_1}{2\pi i} \dots \oint \frac{dz_N}{2\pi i} \frac{(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)}{(z_1 - z_1^{(0)})(z_2 - z_2^{(0)})(z_3 - z_3^{(0)})} \mu(z_1, \dots, z_N) V^{[N]}(z_1, \dots, z_N) |\psi_1\rangle_1 \dots |\psi_N\rangle_N. \end{aligned} \quad (3.1)$$

By Cauchy’s theorem, the pole terms $(z_i - z_i^{(0)})^{-1}$ effectively act as δ -functions fixing three Koba Nielsen variables (z_1, z_2, z_3) to three arbitrarily given points $(z_1^{(0)}, z_2^{(0)}, z_3^{(0)})$ in the complex plane (often taken to be ∞ , 0 and 1.) To maintain the Möbius invariance of the full amplitude the gauge fixing function must be accompanied by a Faddeev Popov determinant. For the chiral string this is the holomorphic square root of the corresponding expression for the closed string (see e.g. [35]), which accounts for the factor $\Delta_{\text{FP}} = (z_1 - z_2)(z_2 - z_3)(z_1 - z_3)$. For the above formula to be unambiguously defined we must in addition specify the integration contours over which

³These variables which carry indices should not be confused with the formal variables z, w, y .

the integrals are to be performed; this will be done in due course. Because the vertex $V^{[N]}$ is not unique, an extra measure factor μ is needed to compensate for the conformal transformations relating different N -vertices to one another, so as to obtain a unique and well-defined scattering amplitude after integration over the Koba Nielsen variables. By construction, this factor is not affected by the global Möbius transformations which act in the same way on all z_i ; it will be explicitly given in the next section.

According to the approach developed in [48, 47, 49, 51, 50], multistring vertices can be characterized in more mathematical terms as follows.

Definition 2

An **N-string vertex (at tree level)** is a multilinear map $V^{[N]} : \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_N \rightarrow \mathbf{C}$, where each \mathcal{F}_i is isomorphic to the Fock space \mathcal{F} of the free string, depending on N complex parameters z_1, \dots, z_N , which satisfies the (unintegrated) **overlap identities**

$$V^{[N]} \Phi^{(i)}(\xi_i) = V^{[N]} \Phi^{(j)}(\xi_j) \left(\frac{d\xi_j}{d\xi_i} \right)^h \quad \forall i, j = 1, \dots, n. \quad (3.2)$$

relating the action of a conformal operator $\Phi(\xi)$ of weight h on one external line (i.e. tensor factor) of the vertex to its action on another external line. Here ξ_i denotes a coordinate patch on the Riemann sphere around the Koba Nielsen point z_i , i.e. $\xi_i(z_i) = 0$.

This definition determines the N -vertex only up to multiplication by an arbitrary function $f = f(z_1, \dots, z_N)$, which could be fixed for instance by demanding $V^{[N]}|\mathbf{0}\rangle_1 \dots |\mathbf{0}\rangle_N = 1$. A more physical way of normalizing $V^{[N]}$ is to require the string scattering amplitudes to be unitary and thereby also to fix the measure [47, 49].

Definition 3

A **physical string scattering amplitude** is a multilinear map $W : \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_N \rightarrow \mathbf{C}$ constructed by means of (3.1) from an N -vertex satisfying **unitarity**, i.e. null physical states should decouple from the S -matrix. Thus for physical states ψ_1, \dots, ψ_N , we require

$$\exists i : \langle \psi_i | \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{P}^1 \quad \implies \quad W(\psi_1, \dots, \psi_N) = 0. \quad (3.3)$$

These definitions, and especially the overlap identities with their implications will be discussed in the following section. Here we only make some preliminary remarks. Since $V^{[N]}$ is only determined up to a normalization, we are free to absorb μ and the Faddeev Popov determinant into it and will frequently do so. It should be clear that the overlap equations are defined only for overlapping charts, but since the Riemann surface appropriate for tree level scattering of strings is the Riemann sphere and this surface only needs two coordinate patches to cover it, the coordinates ξ_i will be defined everywhere except for one point. However, Eq. (3.2) is only valid if the action of $\Phi^{(i)}(\xi_i)$ on the vertex converges, this will not be the case as ξ_i approaches any of the Koba Nielsen points $z_j, j \neq i$. Rigorously speaking, the overlap identities should be understood in the sense of analytic continuation of matrix elements. Interpreted as formal series, they look somewhat different. In Sect. 4 we will establish for the three-vertex the precise relation between the overlaps and the Jacobi identity for intertwining operators. In the appendix we will demonstrate how the unitarity condition leads to set of first order differential equations which determine the measure μ .

Let us now explain our notation. Where appropriate we will suppress the dependence of the vertex on the Koba Nielsen points z_i . For the N -vertex $V^{[N]}$ we will also sometimes write $V_{k_1 \dots k_N}$, explicitly exhibiting the numbering of the external legs. The ket notation $|\psi_i\rangle_k$ denotes the state $\psi_i \in \mathcal{F}$ as an element of the N -string Fock space, i.e. as living in the k -th tensor factor, \mathcal{F}_k . Upper indices (k) on an operator \mathcal{O} indicate the tensor factor, \mathcal{F}_k , on which the operator acts, i.e. $\mathcal{O}^{(k)} = \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes \mathcal{O} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}$ with the operator \mathcal{O} in the k -th place.

Starting from the above intrinsic definition of a multistring vertex we shall see that in the usual oscillator representation the vertex will be of the form

$$V^{[N]} = \langle \tilde{\mathbf{0}} | \mathcal{O}(\{\alpha_m^{(i)\mu}\}), \quad (3.4)$$

with

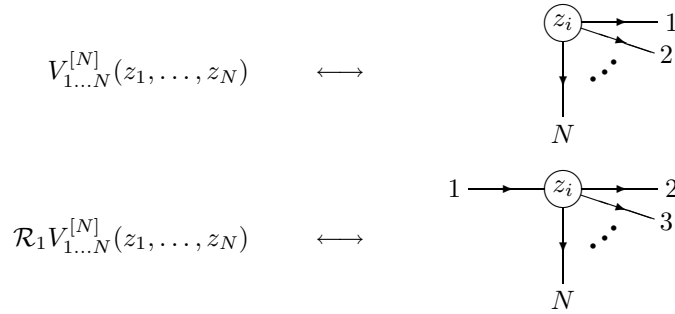
$$|\tilde{\mathbf{0}}\rangle := \sum_{\substack{\mathbf{r}_i \in \Lambda \quad \forall i \\ \mathbf{r}_1 + \dots + \mathbf{r}_N = \mathbf{0}}} |\mathbf{r}_1\rangle \otimes \dots \otimes |\mathbf{r}_N\rangle, \quad (3.5)$$

for some operator \mathcal{O} . Note that $\mathbf{r}_1 + \dots + \mathbf{r}_N = \mathbf{0}$ is just momentum conservation and that the state $\langle \tilde{\mathbf{0}} |$ is not normalizable (even for discrete momenta). \mathcal{O} will turn out to be an exponential of bilinears in the oscillators. Due to the Fourier transform of the bra vacuum, $\langle \tilde{\mathbf{0}} |$, in front of \mathcal{O} it is clear that no creation operators, α_m^μ for $m < 0$, can occur and thus no normal-ordering is required. It will also be important later that our definition of the bra-states in (3.5) includes the cocycle factors $c_{\mathbf{r}_i}$ (cf. Eq. (2.64) and the discussion following it), and this prescription will automatically take care of unwanted factors of (-1) in the commutator of multistring vertices.

We will also need the mathematical prescription for turning around a leg of the vertex. This is essentially provided by the isomorphism \mathcal{R} of Sect. 2.1 which identifies the one-string Fock space \mathcal{F} with its restricted dual space, \mathcal{F}' . Recall that \mathcal{R} is defined as $\mathcal{R} : |\mathbf{r}\rangle \mapsto -\langle -\mathbf{r} | \equiv (\mathbf{r} |_-$, $\alpha_{-m}^\mu \mapsto -\alpha_m^\mu$, so that $\langle\langle \mathcal{R}(\psi) | \varphi \rangle\rangle = (\psi | \varphi) = -(\theta(\psi) | \varphi)$ by (2.91) and (2.98). Then the **reversing operator**, \mathcal{R}_i , which turns leg i around acts, in the oscillator representation, on the N -vertex as follows: it leaves the tensor factors \mathcal{F}_k for $k \neq i$ invariant and maps \mathcal{F}_i to $\mathcal{R}_i(\mathcal{F}_i) = \mathcal{F}'_i$, i.e.

$$\begin{aligned} \mathcal{R}_i : \alpha_m^{(k)\mu} &\mapsto \alpha_m^{(k)\mu} \quad \text{for } k \neq i, \\ \alpha_n^{(i)\nu} &\mapsto -\alpha_{-n}^{(i)\nu}, \\ \langle \tilde{\mathbf{0}} | \mathcal{O} &\mapsto - \sum_{\substack{\mathbf{r}_j \in \Lambda \quad \forall j \\ \mathbf{r}_1 + \dots + \mathbf{r}_N = \mathbf{0}}} {}_1\langle \mathbf{r}_1 | \dots {}_{i-1}\langle \mathbf{r}_{i-1} | {}_{i+1}\langle \mathbf{r}_{i+1} | \dots {}_N\langle \mathbf{r}_N | \mathcal{R}_i \mathcal{O} | -\mathbf{r}_i \rangle_i. \end{aligned} \quad (3.6)$$

The above notational conventions can be most conveniently summarized by the diagrams below:



where the arrows on the legs of the vertex $V^{[N]}$ by definition point outwards and only those point inwards which are turned around.

3.2 Overlap identities

As we will not consider loop amplitudes in this paper, the worldsheet of the string is always taken to be the complex plane (alias the Riemann sphere), which we parametrize by the complex variable ζ in the region of its south pole and $\tilde{\zeta} = \zeta^{-1}$ in the region of its north pole. The N -vertex $V^{[N]}$ quite generally describes the scattering of N string states, which are emitted or absorbed at the Koba-Nielsen points $z_i \in \mathbf{C}$ ($i = 1, \dots, N$). Hence the vertex will depend on the variables z_i . As already pointed out, these must be integrated over before one obtains the final string scattering amplitude. Each z_i belongs to a coordinate neighbourhood parametrized by a complex analytic coordinate $\xi_i = \xi_i(\zeta)$, meaning an analytic function of ζ in the neighbourhood of z_i , which vanishes at the corresponding Koba Nielsen point, i.e. $\xi_i(z_i) = 0$. Apart from this restriction, the functions $\xi_i(\zeta)$ can be chosen arbitrarily; this freedom accounts for the multitude of N -string vertices. We will often identify the variable ζ with one of the coordinates, viz. $\xi_2 \equiv \zeta$. In principle, all of the coordinate patches can be extended to cover the whole complex plane, with the exception of one point, and we will implicitly make use of this fact below in assuming that any two given patches can be made to overlap. The point here is that when the string, i.e. $\Phi^{(i)}(\xi_i)$, acts on the vertex, the result can be expanded in a Laurent series around $\xi_i = 0$; when ξ_i approaches any other Koba Nielsen point, this series will diverge in general. So one should think of the neighbourhoods surrounding the Koba Nielsen points as defining the domains of convergence of the relevant series rather than coordinate patches in the sense of differentiable manifolds.

On the domain where they overlap, the local coordinates ξ_i and ξ_j are related by transition functions τ_{ij} such that

$$\xi_i = \xi_i(\xi_j) = \tau_{ij}(\xi_j). \quad (3.7)$$

If, for example, $\zeta \equiv \xi_2$ we have $\xi_i = \tau_{i2}(\zeta)$, and then the Koba Nielsen variables are given by $z_i = \tau_{2i}(0)$. The transition functions obey the self-evident relations

$$\begin{aligned}\tau_{ii} &= \text{id}, \\ \tau_{ij}\tau_{ji} &= \text{id}, \\ \tau_{ij}\tau_{jk} &= \tau_{ik}\end{aligned}\tag{3.8}$$

(the second identity obviously follows from the other two). Let $\Phi^{(i)}(\xi_i)$ be a conformal field of weight h associated with the i -th Fock space \mathcal{F}_i and the Koba Nielsen point z_i . The fundamental (“unintegrated”) overlap identity (3.2) defining the vertex can be rewritten in terms of differentials as

$$V^{[N]}\Phi^{(i)}(\xi_i(\zeta)) [d\xi_i(\zeta)]^h = V^{[N]}\Phi^{(j)}(\xi_j(\zeta)) [d\xi_j(\zeta)]^h.\tag{3.9}$$

At a superficial glance this might just be regarded as expressing the transformation properties of a differential of weight h on the complex plane under analytic coordinate transformations. However, the content of the above equation is far more subtle. All the difference is made by the superscripts on the conformal fields indicating to which string Fock space they belong. The overlap identities thus relate *different* one-string Fock spaces in a highly non-trivial manner.

Although (3.9) is the most general form of the overlap identities, the so-called integrated overlap identities are sometimes even more useful in practical calculations. To derive them we choose an arbitrary function $f(\xi_i)$, which is analytic except at $\xi_i = 0$ where it may have a pole, and integration contours \mathcal{C}_j surrounding the Koba Nielsen points $\zeta = z_j$, i.e. $\xi_j = 0$. Deforming the contour \mathcal{C}_i and keeping in mind that the poles at $\zeta = z_j$ for $j \neq i$ in general prevent us from pulling it over the other Koba Nielsen points away to infinity, we get

$$V^{[N]} \oint_{\mathcal{C}_i} d\xi_i(\zeta) \Phi^{(i)}(\xi_i(\zeta)) f(\xi_i(\zeta)) = - \sum_{j \neq i} V^{[N]} \oint_{\mathcal{C}_j} d\xi_i(\zeta) \Phi^{(i)}(\xi_i(\zeta)) f(\xi_i(\zeta)).\tag{3.10}$$

The non-trivial step is now to apply the unintegrated overlap equation (3.9) to this expression to obtain the integrated overlap equations. This step is required in order to maintain convergence as discussed above. Now taking the coordinates ξ_j as integration variables we find [48, 47, 49]

$$V^{[N]} \sum_{j=1}^N \oint_{\xi_j=0} d\xi_j \Phi^{(j)}(\xi_j) \left[\frac{d\xi_j(\xi_i)}{d\xi_i} \Big|_{\xi_i=\xi_i(\xi_j)} \right]^{h-1} f(\xi_i(\xi_j)) = 0.\tag{3.11}$$

Of course, this identity is valid for all i .

For each set of transitions functions $\{\tau_{ij}\}$ and Koba Nielsen variables $\{z_i\}$ one can explicitly construct a unique N -vertex associated with them. It is most compactly represented in the form

$$V^{[N]}(\{z_i\}, \{\tau_{ij}\}) = \langle \tilde{0} | \exp \left\{ - \frac{i}{2} \sum'_{i,j} \oint_{\xi_i=0} d\xi_i \mathbf{P}^{(i)}(\xi_i) \cdot \mathbf{X}_{>}^{(j)}(\xi_j(\xi_i)) \right\} \mathcal{N}(\{\boldsymbol{\alpha}_0^{(j)}\})\tag{3.12}$$

where the prime on the sum indicates that we sum over all i, j with $i \neq j$ and $\mathbf{P} \cdot \mathbf{X} \equiv P^\mu X_\mu$ with the Fubini–Veneziano fields defined as in Eqs. (2.69) and (2.72). The zero mode part in (3.12) is given by (cf. [51])

$$\mathcal{N}(\{\boldsymbol{\alpha}_0^{(j)}\}) := \prod_{i < j} \left[\frac{d}{d\xi} (\Gamma \circ \tau_{ij})(\xi) \Big|_{\xi=0} \right]^{-\frac{1}{2} \mathbf{P}^{(i)} \cdot \mathbf{P}^{(j)}}.\tag{3.13}$$

To prove the general formula (3.12) one could use the unintegrated overlap formula (3.2) for the conformal field $\Phi = \mathbf{X}$; however, it is simpler to first check its non-zero mode part by means of the integrated overlap conditions (3.11) and then the zero mode part by (3.9). For this purpose we need to expand powers of the transition functions as follows (these expansions were already introduced in [1])

$$\frac{1}{\sqrt{m}} \left\{ [\tau(\xi)]^m - [\tau(0)]^m \right\} = \sum_{n \geq 1} C_{mn}(\tau) \frac{\xi^n}{\sqrt{n}},\tag{3.14}$$

where τ stands for any τ_{ij} . Substituting $\Phi^{(i)}(\xi_i) = \mathbf{P}^{(i)}(\xi_i)$ (which is of weight one, i.e. $h = 1$) into the integrated overlap (3.11) with the complete set of allowable functions $f(\xi_i) = (\xi_i)^{-m}$ and making use of the above expansions, we immediately deduce the overlap equations for the oscillators

$$V^{[N]} \left\{ \alpha_{-m}^{(i)\mu} + \sqrt{m} \sum_{j \neq i} \sum_{n \geq 1} C_{mn}^{ij} \frac{\alpha_n^{(j)\mu}}{\sqrt{n}} + \sum_{j \neq i} [(\Gamma \circ \tau_{ij})(0)]^m \alpha_0^{(j)\mu} \right\} = 0, \quad (3.15)$$

where we have defined

$$C_{mn}^{ij} := C_{mn}(\Gamma \circ \tau_{ij}), \quad (3.16)$$

with

$$\Gamma(\xi) := \frac{1}{\xi}. \quad (3.17)$$

Note that $C_{mn}^{ij} = C_{nm}^{ji}$, i.e. $(C^{ij})^T = C^{ji}$ for the matrices⁴. It is then not difficult to check that the following expression in terms of oscillators satisfies Eq. (3.15):

$$V^{[N]} = \langle \tilde{\mathbf{0}} | \exp \left\{ -\frac{1}{2} \sum'_{i,j} \sum_{m,n \geq 1} \frac{\alpha_m^{(i)\mu}}{\sqrt{m}} C_{mn}^{ij} \frac{\alpha_n^{(j)\mu}}{\sqrt{n}} - \sum'_{i,j} \sum_{m \geq 1} \frac{\alpha_m^{(i)\mu}}{m} [(\Gamma \circ \tau_{ij})(0)]^m \alpha_{0\mu}^{(j)} + \ln \mathcal{N}(\alpha_0^{(i)}) \right\} \quad (3.18)$$

with an as yet undetermined function \mathcal{N} of the zero mode oscillators. To fix the latter terms we have to make use of the unintegrated overlap with $\Phi^{(i)}(\xi_i) = \mathbf{X}^{(i)}(\xi_i)$. By expanding the Fubini–Veneziano fields in (3.12) in terms of oscillators it is now straightforward to verify that the non-zero mode part of (3.12) indeed coincides with the above formula (3.18) for the N vertex. We stress once more that (3.18) can still be multiplied by an arbitrary function without affecting the overlap equations.

Finally, we have to spell out the measure μ occurring in our general formula (3.1). Relegating the details of the derivation to Appendix A, where also some examples are worked out, we just quote the result:

$$\mu(z_1, \dots, z_N) = \prod_{j=1}^N \frac{\partial \xi_j(\zeta)}{\partial \zeta} \Big|_{\zeta=z_j}. \quad (3.19)$$

In the “gauge” $\zeta = \xi_1$, this expression becomes

$$\mu(z_1, \dots, z_N) = \prod_{j=2}^N \frac{\partial \tau_{j1}(\xi_j)}{\partial \xi_j} \Big|_{\xi_j=0}. \quad (3.20)$$

From (3.20) one infers that the measure $\mu(z_1, \dots, z_N)$ is insensitive to those variable transformations which leave the transition functions (and hence the vertex) invariant. We will see at the end of this section (cf. (3.31) below) that there is one such conformal transformation which acts in the same manner on all z_i .

To demonstrate the power of the integrated overlap equations let us consider the example of primary fields of weight $h = 1$, i.e. vertex operators associated with physical states. The choice $f \equiv 1$ in Eq. (3.11) immediately gives

$$V^{[N]} \sum_{j=1}^N \oint_{\xi_j=0} d\xi_j \Phi^{(j)}(\xi_j) = 0. \quad (3.21)$$

An important special case of this formula, which we will return to in Sect. 5, is obtained by taking the weight one primary fields to be transversal and longitudinal DDF operators; the identity tells us that these can just be moved through the vertex without any change or extra contributions. Note that the form of the overlap equations for weight one primary fields is universally valid for arbitrary N -vertices whereas the overlap equations for other conformal weights will explicitly depend on the choice of transition functions.

The overlap identities also make sense for objects which transform in an anomalous but still well defined way under conformal transformations. An example is the energy momentum tensor \mathcal{T} for which $\tilde{\mathcal{T}}(\tilde{\xi}) =$

⁴ If τ is a Möbius function, we have $C(\tau)^T = C(\Gamma \circ \tau^{-1} \circ \Gamma)$.

$\mathcal{T}(\tilde{\xi}) \left(\frac{d\tilde{\xi}}{d\xi} \right)^2 + \frac{c}{12} (S\tilde{\xi})(\xi)$, where $(Sf)(\xi) = \frac{f'''(\xi)}{f'(\xi)} - \frac{3}{2} \left(\frac{f''(\xi)}{f'(\xi)} \right)^2$ is the Schwarzian derivative and $c = 26 - d$ the conformal anomaly. Hence apart from the last term, the energy momentum tensor is of weight two. The associated (unintegrated) overlap identity is

$$V^{[N]} \mathcal{T}^{(i)}(\xi_i) = V^{[N]} \mathcal{T}^{(j)}(\xi_j) \left(\frac{d\xi_j}{d\xi_i} \right)^2 + \frac{c}{12} V^{[N]} (S\xi_j)(\xi_i) \quad \forall i, j = 1, \dots, n. \quad (3.22)$$

Just as before we can derive a corresponding integrated overlap equation for the Virasoro generators from the above unintegrated equation. In this process one would seem to acquire additional contributions from the Schwarzian. However, for tree level vertices these contributions vanish. This can be shown either directly by taking the integrated overlaps and saturating all legs with the true vacuum, using the explicit form of the vertex given above, or by explicitly performing the integrations over the ξ_j 's which vanish as the Schwarzian term contains no relevant poles. This is what we should also expect on physical grounds without any calculation because the central term in the Virasoro algebra giving rise to the Schwarzian is a quantum effect (carrying a factor of \hbar) and thus not visible at string tree level. As a result we find

$$V^{[N]} \sum_{j=1}^N \oint_{\xi_j=0} d\xi_j \mathcal{T}^{(j)}(\xi_j) \frac{d\tau_{ji}(\xi_i)}{d\xi_i} \Big|_{\xi_i=\tau_{ij}(\xi_j)} f(\tau_{ij}(\xi_j)) = 0. \quad (3.23)$$

Of course, in a string loop expansion one does encounter additional contributions to the integrated overlap equation for the energy momentum tensor at higher orders in \hbar (unless $d = 26$), and these play a very important role in the determination of string loop corrections to $V^{[N]}$.

Next we study the conformal properties of the N -vertex; the freedom of performing such transformations is considerably greater than for one-string vertex operators and constitutes one of the main advantages of the multistring formalism. If \mathcal{M} is a conformal transformation $\xi \rightarrow \mathcal{M}(\xi)$, we denote by $\hat{\mathcal{M}}$ the realization of the *same* conformal transformation as an operator on Fock space. In general this conformal transformation need not be analytic at $\xi = 0$. As before, a superscript indicates on which Fock space the transformation acts. For instance, $\hat{\mathcal{M}}_i^{(j)}$ corresponds to the realization of the conformal transformation \mathcal{M}_i on the j -th Fock space \mathcal{F}_j . Explicitly,

$$\hat{\mathcal{M}}_i^{(j)} = \exp \left\{ \sum_{n=-\infty}^{\infty} c_n^i L_n^{(j)} \right\}, \quad (3.24)$$

where the c_n^i are arbitrary parameters. Of course, this expression has to be taken with a grain of salt as the Virasoro algebra cannot be globally exponentiated to a group (see e.g. [2] for a discussion of this point). We can then define a new N -vertex $\tilde{V}^{[N]}$ from $V^{[N]}$ by

$$\tilde{V}^{[N]} \equiv V^{[N]}(\{\tilde{z}_i\}, \{\tilde{\tau}_{ij}\}) := V^{[N]}(\{z_i\}, \{\tau_{ij}\}) \prod_{j=1}^N \hat{\mathcal{M}}_j^{(j)}, \quad (3.25)$$

where we can choose the transformations \mathcal{M}_i independently on each leg of the vertex. If the conformal transformations $\mathcal{M}_i(\xi)$ are analytic at $\xi = 0$ then one finds that one obtains the same scattering amplitude and due to this freedom there is an enormous variety of N -string vertices which has no counterpart in terms of one-string vertices. One way to see this result is that if the transformation is analytic at $\xi = 0$ then the sum in (3.24) begins with $n = -1$ and when saturated with physical states, those terms in the sum which contain $L_n, n \geq 1$ annihilate on these states. The terms with L_0 possibly give functions of z_i if the conformal transformation depends on these, and the L_{-1} 's implement the shift of the Koba-Nielsen variables, discussed below, in the vertex. Once one takes into account the calculation of the measure using null state decoupling then one finds that it changes so as to compensate for the latter two effects and the scattering amplitude is the same.

The conformal transformations \mathcal{M}_i will also shift the Koba Nielsen points and yield new transition functions. If the new coordinates $\tilde{\xi}_i$ are given by

$$\tilde{\xi}_i = \mathcal{M}_i^{-1}(\xi_i), \quad (3.26)$$

the Koba Nielsen variables are transformed according to

$$\tilde{z}_i = (\xi_i^{-1} \circ \mathcal{M}_i \circ \xi_i)(z_i). \quad (3.27)$$

The relation between the new transition functions and the old ones is

$$\tilde{\tau}_{ij} = \mathcal{M}_i^{-1} \circ \tau_{ij} \circ \mathcal{M}_j. \quad (3.28)$$

Next we recall from Sect. 2.1 that a conformal field of weight h transforms as follows under a conformal mapping \mathcal{M} :

$$\hat{\mathcal{M}}^{-1}\Phi(\xi)\hat{\mathcal{M}} = \left[\frac{d\mathcal{M}^{-1}(\xi)}{d\xi} \right]^h \Phi(\mathcal{M}^{-1}(\xi)). \quad (3.29)$$

The covariance of the overlap equations can be inferred from

$$V^{[N]}\Phi^{(i)}(\xi_i) \cdot \prod_{k=1}^N \hat{\mathcal{M}}_k^{(k)} = \tilde{V}^{[N]}\Phi^{(i)}(\mathcal{M}_i^{-1}(\xi_i)) \left[\frac{d\mathcal{M}_i^{-1}(\xi_i)}{d\xi_i} \right]^h. \quad (3.30)$$

Namely the new vertex $\tilde{V}^{[N]}$ then obeys the overlap condition (3.9) with $\Phi^{(i)}$ replaced by the transformed fields (3.29) (for each i) and the new transition functions (3.28).

Let us determine the conformal transformations which leave a given vertex inert. If we demand $\tilde{\tau}_{ij} = \tau_{ij}$ for all i, j , then Eq. (3.28) implies

$$\mathcal{M}_j = \tau_{ji} \circ \mathcal{M}_i \circ \tau_{ij}. \quad (3.31)$$

This shows that, given a vertex, we always have the freedom to apply an arbitrary conformal transformation \mathcal{M}_i to a certain leg without changing the vertex as long as we compensate by conformal transformations on the other legs according to the above formula; this will turn out to be useful later. Even though the vertex remains invariant, however, the Koba Nielsen variables will be shifted by virtue of (3.27). Using (3.31), it is easy to see that in this case we have $\xi_i^{-1} \circ \mathcal{M}_i \circ \xi_i = \xi_j^{-1} \circ \mathcal{M}_j \circ \xi_j$ for all i, j , and therefore the conformal transformation acts *in the same way* on all Koba Nielsen variables. For global and univalent mappings of the Riemann sphere onto itself the freedom is consequently reduced to one Möbius transformation, which we identify with the transformation used at the beginning of this chapter to gauge-fix three Koba Nielsen points to arbitrary non-coincident points. We stress again that, while the on-shell vertices are all equivalent to each other under arbitrary conformal transformations, here we have a degree of freedom for the off-shell vertices.

Finally, we note that if $\hat{\mathcal{M}}$ has conformal transformation \mathcal{M} then $\hat{\mathcal{M}}^\dagger$ (or $\hat{\mathcal{M}}^\#$) has conformal transformation $\Gamma \circ \mathcal{M}^{-1} \circ \Gamma$ as can be seen by taking the adjoint of Eq. (3.29) for $\Phi = \mathbf{X}$ (so that $h = 0$) and using relation (2.102). Thus

$$\hat{\mathcal{M}}^\# = \hat{\mathcal{M}}^\dagger = \Gamma \widehat{\mathcal{M}^{-1}} \Gamma. \quad (3.32)$$

4 Vertex operators and multistring vertices

Having set up the multistring formalism in general terms, we now wish to relate it to the one-string formulation of Sect. 2. As we have already mentioned, the correspondence is not one-to-one as there are many more multistring vertices than one-string vertex operators; of course the final answer for the computation of physical scattering amplitude is always the same, as the vertices differ only off shell. We will also explain in this chapter how multistring vertices can be constructed from three-vertices by sewing, and how explicit expressions for them can be obtained quickly and efficiently by means of overlap identities.

4.1 Vertex operators from three-vertices

Given any three-string vertex $V^{[3]}(z_1, z_2, z_3)$, derived from overlap equations, we can define a linear map $\mathcal{V} : \mathcal{F} \rightarrow (\text{End } \mathcal{F})[[z_1, z_1^{-1}, z_2, z_2^{-1}, z_3, z_3^{-1}]]$, which assigns to each state $\psi \in \mathcal{F}$ a “vertex operator” $\mathcal{V}(\psi; z_1, z_2, z_3)$ defined by its action on all states $\varphi \in \mathcal{F}$:

$$\mathcal{V}(\psi; z_1, z_2, z_3)\varphi := \mathcal{R}_1 \left[V^{[3]}(z_1, z_2, z_3) |\varphi\rangle_2 |\psi\rangle_3 \right]. \quad (4.1)$$

Note that such an operator in general is a formal series in the Koba Nielsen variables. Out of this huge number of vertex operators, however, only a certain class leads to physically acceptable operators. We shall see that the requirement of mutual locality of vertex operators is equivalent to the postulate $\tau_{12} = \Gamma$ which in turn

implies $z_1 = \infty$ and $z_2 = 0$ if we put $\zeta \equiv \xi_2$. Imposing the physical requirement that operators associated with physical states should be primary fields of weight one w.r.t. the variable z_3 , we will be led to the same choice for τ_{12} . Therefore, rather than using the most general three-vertex for the definition of vertex operators, we will in this section deal with a class of vertices $V^{[3]}(z) \equiv V^{[3]}(\infty, 0, z)$ which depend on the third Koba Nielsen variable $z_3 \equiv z$ as a parameter while z_1 and z_2 are fixed. Consequently, the fundamental identity relating such a three-string vertex $V^{[3]}(z)$ to a one-string vertex operator $\mathcal{V}(\psi, z)$ is

$$\mathcal{V}(\psi, z)\varphi := \mathcal{R}_1 \left[V^{[3]}(z) |\varphi\rangle_2 |\psi\rangle_3 \right]. \quad (4.2)$$

Remarkably, the operator $\mathcal{V}(\psi, z)$ is automatically normal-ordered since its action on arbitrary Fock space elements is always well-defined by the right-hand side. Somewhat loosely we can also write⁵

$$\mathcal{V}(\psi, z) := \mathcal{R}_1 \left[V^{[3]}(z) |\psi\rangle_3 \right], \quad (4.3)$$

where equality is to be understood in the sense of matrix elements. It is immediately clear that the correspondence between three-vertices $V^{[3]}(z)$ and vertex operators $\mathcal{V}(\psi, z)$ (for all ψ) cannot be one-to-one, for we can always multiply the vertex $V^{[3]}(z)$ by conformal transformations of type (3.31) which leave the vertex inert and thus do not alter the left-hand side of (4.3). We should also emphasize that these three-vertices do not in general lead to vertex operators which satisfy all the axioms of a vertex algebra, but nonetheless can be used to construct the Lie algebra of physical states since on-shell the vertices are all the same. The only difference is made by their off-shell properties. In Sect. 4.3 we will give two explicit examples of such vertices, one of which does lead via the above definition to the vertex operator we have been working with in Sect. 2. In fact, it will be much easier to prove the axioms of a vertex algebra than to produce the explicit expressions for the vertex operators in the oscillator representation.

Symbolically, we have

$$\mathcal{V}(\psi, z) \longleftrightarrow \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ |\psi\rangle_3 \end{array}$$

where normal-ordering is already built by the above argument; we will later check this explicitly for the tachyon emission vertex.

Let us now turn to the discussion of the physical requirements which truncate the family of three-vertices. First, we demand that for any physical state the corresponding vertex operator defined by Eq. (4.3) is a primary field of weight one. As already mentioned, in this case the integrated physical vertex operator $\psi_0 \equiv \text{Res}_z [\mathcal{V}(\psi, z)]$ commutes with the Virasoro generators and maps physical states into physical states. It is then straightforward to show that under these circumstances null states decouple from the three-vertex if all legs are saturated with physical states, viz.

$$\begin{aligned} \text{Res}_z \left[V^{[3]}(z) L_{-n}^{(1)} |\chi\rangle_1 |\varphi\rangle_2 |\psi\rangle_3 \right] &= \left(L_{-n} \chi \mid \text{Res}_z [\mathcal{V}(\psi, z)] \varphi \right) \\ &= \left(\chi \mid [L_n, \text{Res}_z [\mathcal{V}(\psi, z)]] \varphi \right) + \left(\chi \mid \text{Res}_z [\mathcal{V}(\psi, z)] L_n \varphi \right) \\ &= 0. \end{aligned} \quad (4.4)$$

Cyclicity of the three-vertex (see Sect. 5.1 below) guarantees that this decoupling of spurious (and in particular null physical) states holds for any leg.

Translated into the framework of three-string vertices the condition (2.20) for $h = 1$ reads

$$V^{[3]}(z) \left\{ -L_n^{(2)} + L_{-n}^{(1)} - (n+1)z^n \right\} - z^{n+1} \frac{d}{dz} V^{[3]}(z) = 0. \quad (4.5)$$

We note that this condition is sufficiently strong to also fix the overall normalization of the vertex, as the last term is clearly sensitive to multiplication of $V^{[3]}$ by a function of z . We can relax this condition by combining it with the condition for $n = 0$, which permits us to swap the $\frac{d}{dz}$ for L_0 terms. The z^n term can only come from

⁵A specific example of this equation relating one-string vertex operators and three-vertices was given in [62].

$L_0^{(3)}$ acting on $|\psi\rangle_3$, however we may also have L_n ($n \geq 1$) terms in the corresponding overlap; they annihilate on ψ as it is physical. Thus if we require

$$V^{[3]}(z) \left\{ -L_n^{(2)} + L_{-n}^{(1)} + z^n (L_0^{(2)} - L_0^{(1)}) - n z^n L_0^{(3)} + \text{terms in } L_m^{(3)}, m \geq 1 \right\} = 0, \quad (4.6)$$

instead of (4.5), we get a much larger class of three-vertices whose normalization is no longer fixed. In order to work out the ensuing restrictions on the transition functions we must compare Eq. (4.6) with the overlap equation (3.23) for the energy momentum tensor. Hence, if we choose $i = 1$ and $f(\xi_1) = (\xi_1)^{1-n} - z^n \xi_1$, then to get the above equation we demand

$$\begin{aligned} f(\xi_1) \frac{d\xi_2}{d\xi_1} &= -(\xi_2)^{n+1} + z^n \xi_2, \\ f(\xi_1) \frac{d\xi_3}{d\xi_1} &= -n z^n \xi_3 + \mathcal{O}((\xi_3)^2). \end{aligned} \quad (4.7)$$

The first equation implies that $\xi_1 = (\xi_2)^{-1}$, i.e.

$$\tau_{12} = \Gamma; \quad (4.8)$$

from the second equation we deduce that f vanishes at $\xi_3 = 0$ which in turn means that $\xi_1(z_3) = z^{-1}$ or $\xi_2(z_3) = z$. Hence if we choose ξ_2 as the coordinate ζ , then $V^{[3]}(z)$ has Koba Nielsen points ∞ , 0 and z for legs 1, 2 and 3, respectively. Since the parameter z just the Koba Nielsen point on leg 3, the vertex operator $\mathcal{V}(\psi, z)$ defined by (4.3) can be interpreted as describing the emission of a physical state ψ from the string at the third Koba Nielsen point $z_3 = z$.

The above analysis shows how an essential physical assumption immediately fixes the transition function τ_{12} and constrains τ_{13} very much. On the other hand, we emphasize that the weight one postulate is somewhat stronger than decoupling of zero norm physical states but with this requirement decoupling is manifest and it will lead us to a class of vertices that are sufficient for our purposes.

We also demand the following creation property (cf. Eq. (2.17)) of the three-vertex:

$$\lim_{z \rightarrow 0} \mathcal{V}(\psi, z) |\mathbf{0}\rangle = \psi. \quad (4.9)$$

This property is equivalent to

$$\begin{aligned} \lim_{z \rightarrow 0} \mathcal{R}_1 V_{123}(z) |\mathbf{0}\rangle_2 \alpha_{-m}^{(3)} &\stackrel{!}{=} \alpha_{-m}^{(1)} \lim_{z \rightarrow 0} \mathcal{R}_1 V_{123}(z) |\mathbf{0}\rangle_2 \\ &= \lim_{z \rightarrow 0} \mathcal{R}_1 \left[V_{123}(z) |\mathbf{0}\rangle_2 \left(\alpha_{-m}^{(1)} \right)^\sharp \right], \end{aligned} \quad (4.10)$$

i.e. ,

$$\lim_{z \rightarrow 0} V_{123}(z) |\mathbf{0}\rangle_2 \left\{ \alpha_{-m}^{(3)} + \alpha_m^{(1)} \right\} \stackrel{!}{=} 0. \quad (4.11)$$

Comparing with the general overlap equations for the oscillators (3.15),

$$V_{123}(z) |\mathbf{0}\rangle_2 \left\{ \alpha_{-m}^{(3)} + \sqrt{m} \sum_{n \geq 1} C_{mn}^{31} \frac{\alpha_n^{(1)}}{\sqrt{n}} + [(\Gamma \circ \tau_{31})(0)]^m \alpha_0^{(1)} \right\} = 0, \quad (4.12)$$

we conclude that

$$\lim_{z \rightarrow 0} C_{mn}^{31}(z) = \delta_{mn} \quad \text{and} \quad \lim_{z \rightarrow 0} [\Gamma \circ \tau_{31}(z; -)](0) = 0, \quad (4.13)$$

i.e. (by (3.16)),

$$\tau_{31}(z; \xi) = \frac{1}{\xi} + \sum_{n \geq 1} f_n(\xi) z^n, \quad (4.14)$$

where the f_n 's denote some functions which are not determined by the creation property.

We now consider how some of the axioms of the vertex algebra approach imply restrictions for the three-vertex via Eq. (4.3). We begin with the vacuum axiom (2.4) which will be certainly fulfilled if we postulate that $\Pi_{12} := \mathcal{R}_1 V_{123}(z)|\mathbf{0}\rangle_3$ is the natural isomorphism which identifies the Fock spaces \mathcal{F}_2 and \mathcal{F}_1 :

$$\Pi_{12} : \mathcal{F}_2 \rightarrow \mathcal{F}_1, |\psi\rangle_2 \mapsto |\psi\rangle_1. \quad (4.15)$$

This means that we require

$$\mathcal{R}_1 V_{123}(z)|\mathbf{0}\rangle_3 \alpha_{-m}^{(2)} \stackrel{!}{=} \alpha_{-m}^{(1)} \mathcal{R}_1 V_{123}(z)|\mathbf{0}\rangle_3, \quad (4.16)$$

i.e. ,

$$V_{123}(z)|\mathbf{0}\rangle_3 \left\{ \alpha_{-m}^{(2)} + \alpha_m^{(1)} \right\} \stackrel{!}{=} 0. \quad (4.17)$$

Observe that this equation is valid for arbitrary z unlike (4.11). Recalling the general overlap equations for the oscillators (3.15),

$$V_{123}(z)|\mathbf{0}\rangle_3 \left\{ \alpha_{-m}^{(2)} + \sqrt{m} \sum_{n \geq 1} C_{mn}^{21} \frac{\alpha_n^{(1)}}{\sqrt{n}} + [(\Gamma \circ \tau_{21})(0)]^m \alpha_0^{(1)} \right\} = 0, \quad (4.18)$$

we conclude that imposing $\tau_{12} = \Gamma$ is equivalent to $\mathcal{V}(|\mathbf{0}\rangle, z) = \text{id}_{\mathcal{F}}$. Happily, we are thus led to the same choice for τ_{12} as above although, somewhat surprisingly, no implications for the other transition functions emerge. In fact, below we shall see that this choice for τ_{12} is quite powerful for it already implies locality for the vertex operators.

As regards the injectivity axiom we recall that its information is already encoded in the above creation property so that we do not get a new condition. Finally, regularity is fulfilled since the vertex contains annihilation operators only and it acts on states with a finite occupation number. Note that the class of three-vertices we have considered so far, do not satisfy all the axioms. We will see that in order to recover the specific vertex operator of Sect. 2 we have to fix all the transition functions.

4.2 Sewing of three-vertices

Originally, N -vertices were computed by sewing three-vertices; the necessary calculations used to be rather cumbersome. We will now describe a much quicker route to their explicit determination by a procedure, the essential steps of which were already given in [50]. All we need to do is to figure out the transition functions for the sewn vertex from the transition functions of the basic three-vertex used in the sewing procedure. Once this is accomplished we must only substitute the results into our “master formula” (3.12) to arrive at the final result.

Sewing of two vertices means that we turn around a leg of the second vertex and glue it together with another leg of the first vertex by taking the string scalar product (after identifying the two Fock spaces, of course). This is symbolically shown in the figure below:

$$V_{123}(z) \stackrel{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) \quad \longleftrightarrow \quad \begin{array}{c} 1 \longrightarrow \textcircled{z} \longrightarrow 2 \cup 2' \longrightarrow \textcircled{w} \longrightarrow 4 \\ \downarrow \qquad \qquad \qquad \downarrow \\ 3 \qquad \qquad \qquad 5 \end{array} \quad (4.19)$$

Note that the notation $\stackrel{ij}{\cup}$ indicates that leg i of the first vertex is sewn with leg j of the second vertex; of course, this only makes sense if either of the legs has been turned around. Under the assumption $\tau_{12} = \Gamma$, the above four-vertex will turn out to be symmetric under the simultaneous interchange of the variables z and w and legs 3 and 5.

As indicated above, we now must only work out the transition functions for the new four-vertex by repeated application of the unintegrated overlap conditions. This means that we act with $\mathbf{X}^{(1)}(\xi_1)$ on leg 1 of the composite vertex and then move it through the vertex by means of the overlap equations. For example,

$$\left[V_{123}(z) \mathbf{X}^{(1)}(\xi_1) \right] \stackrel{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w)$$

$$\begin{aligned}
&= V_{123}(z) \mathbf{X}^{(2)}(\tau_{21}(z; \xi_1)) \overset{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) \\
&= V_{123}(z) \overset{22'}{\cup} \mathbf{X}^{(2')}(\tau_{21}(z; \xi_1)) \mathcal{R}_{2'} V_{2'45}(w) \\
&= V_{123}(z) \overset{22'}{\cup} \mathcal{R}_{2'} \left[V_{2'45}(w) \mathbf{X}^{(2')}([\Gamma \circ \tau_{21}(z)](\xi_1)) \right] \\
&= V_{123}(z) \overset{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) \mathbf{X}^{(5)}([\tau_{52'}(w) \circ \Gamma \circ \tau_{21}(z)](\xi_1)), \tag{4.20}
\end{aligned}$$

where the first entry in $\tau_{ij}(z; \cdot)$ indicates the dependence of the transition function on the parameter, while the second entry is the argument of the function. In the formulas below, we will usually omit the argument and only indicate the parameters explicitly. Note that in the third step we have used the relation $\mathbf{X}^{(i)}(\xi) \mathcal{R}_i V = \mathcal{R}_i [V \mathbf{X}^{(i)\sharp}(\xi)] = \mathcal{R}_i [V \mathbf{X}^{(i)}(\Gamma(\xi))]$ which describes how \mathbf{X} commutes with the reversing operator. We conclude that

$$\tau_{51}(z, w)(\xi) = [\tau_{52'}(w) \circ \Gamma \circ \tau_{21}(z)](\xi). \tag{4.21}$$

The above calculation can be carried over to the determination of the other transition functions of the sewn vertex. The final result can be easily read off from the diagram: the general rule is to compose the basic transition functions appropriately and to keep in mind that every sewing operation \cup is accompanied by the insertion of a factor Γ into the transition function⁶. Repeated application of this method yields the remaining transition functions

$$\tau_{35}(z, w) = \tau_{32}(z) \circ \Gamma \circ \tau_{2'5}(w), \tag{4.22}$$

$$\tau_{14}(z, w) = \tau_{12}(z) \circ \Gamma \circ \tau_{2'4}(w), \tag{4.23}$$

$$\tau_{34}(z, w) = \tau_{32}(z) \circ \Gamma \circ \tau_{2'4}(w). \tag{4.24}$$

where we henceforth suppress the argument ξ . Now, as functions, we have

$$\begin{aligned}
\tau_{12} &= \tau_{2'4}, \\
\tau_{13} &= \tau_{2'5}, \\
\tau_{23} &= \tau_{45}. \tag{4.25}
\end{aligned}$$

If we furthermore take $\tau_{12} = \Gamma$ then the transition functions for the composite vertex simplify to

$$\tau_{14} = \Gamma, \tag{4.26}$$

$$\tau_{15} = \tau_{13}(w), \tag{4.27}$$

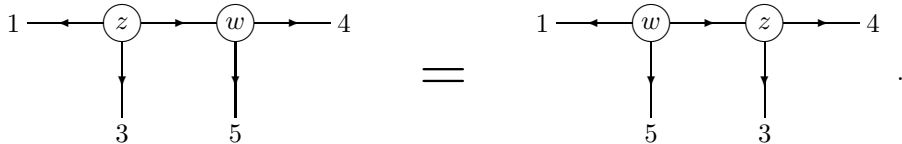
$$\tau_{34} = \tau_{32}(z), \tag{4.28}$$

$$\tau_{35} = \tau_{31}(z) \circ \tau_{13}(w). \tag{4.29}$$

We observe that the transition functions remain the same under the transformation $3 \leftrightarrow 5, z \leftrightarrow w$. For example,

$$\tau_{53}(w, z) = [\tau_{35}(w, z)]^{-1} = [\tau_{31}(w) \circ \tau_{13}(z)]^{-1} = \tau_{31}(z) \circ \tau_{13}(w) = \tau_{35}(z, w).$$

Hence the four-vertex is indeed symmetric under this change; at the level of vertex operators, however, this is just locality of Theorem 1. Pictorially, we have



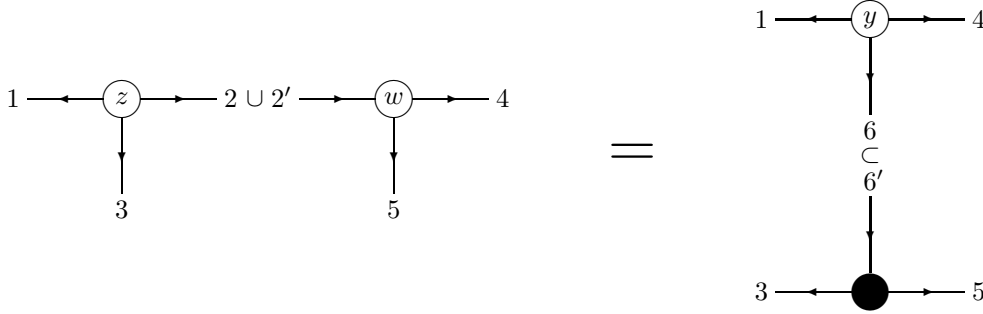
We stress that the overlaps are so powerful that the rather innocent choice $\tau_{12} = \Gamma$ turns out to be equivalent to the principle of locality for vertex operators. But as regards duality for vertex operators in the sense of Theorem 1, we will see later that this requires a specific choice for the other transition functions, too.

⁶At the risk of appearing overly pedantic, we emphasize once more that the insertion of Γ is due to the operation \mathcal{R} , and not to the sewing as such.

In string theory it is often quite useful to sew vertices in different ways. This will allow us to formulate the principle of duality and in our context will be essential for the construction of commutators of integrated physical vertex operators. The relation we would like to focus on next is

$$V_{123}(z) \overset{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) = V_{146}(y) \overset{66'}{\cup} \mathcal{R}_{6'} \bar{V}_{6'53}(z, w, y), \quad (4.30)$$

defining some new vertex $\bar{V}_{6'53}(z, w, y)$. The parameter y on the right-hand side is free and we can choose it as we like. It is by no means obvious that the left-hand side of (4.30) can always be rewritten in the way indicated on the right-hand side; rather this is a very nontrivial property of the underlying physical theory. Pictorially,



To determine the new vertex $\bar{V}^{[3]}$ we use the same techniques as above. Then the new transition functions come out to be

$$\bar{\tau}_{56'} = \tau_{31}(w) \circ \tau_{13}(y) \circ \Gamma, \quad (4.31)$$

$$\bar{\tau}_{36'} = \tau_{31}(z) \circ \tau_{13}(y) \circ \Gamma, \quad (4.32)$$

$$\bar{\tau}_{35} \equiv \tau_{35} = \tau_{31}(z) \circ \tau_{13}(w). \quad (4.33)$$

We observe that $\bar{\tau}_{35}$ does not depend on the free parameter y , whereas for the choice $y = z$ or $y = w$ the transition functions $\bar{\tau}_{36'}$ or $\bar{\tau}_{56'}$, respectively, reduce to Γ . We shall adopt the latter choice from now on, i.e. setting $y = w$, we finally obtain

$$\bar{\tau}_{56'} = \Gamma, \quad (4.34)$$

$$\bar{\tau}_{36'} = \tau_{31}(z) \circ \tau_{13}(w) \circ \Gamma. \quad (4.35)$$

This means that the vertex $\bar{V}_{6'53}(z, w, w)$ is symmetric under the simultaneous interchange of the variable z and w and legs 5 and 6'.

The above calculation together with the diagram can be regarded as the demonstration of string duality for it expresses the same amplitude alternatively as a sum over s -channel poles or t -channel poles. We note, however, that the vertex $\bar{V}^{[3]}$ used to sew with on the right-hand side of the above figure is in general not the same as the vertex $V^{[3]}$ we began sewing with. We will see later that there does exist a specific choice for the transition functions of $V(z)$ such that $\bar{V}^{[3]}$ is equal to $V(z)$, too. Duality in the sense of Theorem 1 is a somewhat more restrictive notion which is only satisfied by the specific vertex alluded to above. Hence the above can be viewed as a generalisation of Theorem 1. The vertex $\bar{V}^{[3]}$ will later prove useful when we analyze the commutator of integrated physical vertex operators.

We now consider how we might construct a general class of three-vertices $V^{[3]}(z)$ with the above properties starting with an arbitrary initial three-vertex $\check{V}^{[3]}$. One advantage of this way of proceeding is that it will provide us with specific examples of three-vertices $V^{[3]}(z)$ and enable us to carry out the above, and other, calculations for them. Although the above formalism is rather elegant, to get a good feel for what is going on it is often best to consider specific examples at least in the first instance. It will also allow us to see the relationship between well-known examples of three-vertices, such as the old CSV vertex [9], and the vertices we use, and to regard the sewing of vertices above in a more traditional manner namely the sewing of arbitrary vertices and propagators.

To be more specific let us start with an arbitrary three-vertex $\check{V}^{[3]}$, which satisfies the overlap equations for some given transition functions $\check{\tau}_{ij}$; neither the vertex nor the transition functions are assumed to depend on the parameter z . Next we must find a suitable $V^{[3]}(z)$ and hence $\bar{V}^{[3]}(z, w, w)$ which by the results of the previous

section is equivalent to the determination of the conformal mappings \mathcal{M}_i relating the vertices to one another. From Eq. (3.28) we know that conformal transformations \mathcal{M}_j exist such that

$$V^{[3]}(z) = \check{V}^{[3]} \prod_j \hat{\mathcal{M}}_j^{(j)}. \quad (4.36)$$

Demanding $\tau_{12} = \Gamma$ implies that $\mathcal{M}_2 = \check{\tau}_{21} \circ \mathcal{M}_1 \circ \Gamma$. We recall that we can always choose one of the conformal transformations at will (cf. Eq. (3.31)). We choose

$$\mathcal{M}_1 \equiv \sigma_z, \quad (4.37)$$

with $\sigma_z(\xi) := z\xi$; the associated transformation $\hat{\mathcal{M}}_1$ is then just the scaling operator $\hat{\sigma}_z = z^{L_0}$. This immediately leads to

$$\mathcal{M}_2 = \check{\tau}_{21} \circ \sigma_z \circ \Gamma. \quad (4.38)$$

We also demand that $\tau_{23}(0) = z$, since the vertex $V^{[3]}(z)$ should have Koba Nielsen point $z_3 = z$ for leg 3 in ξ_2 coordinates. This implies for \mathcal{M}_3

$$\mathcal{M}_3(0) = \check{\tau}_{31}(1). \quad (4.39)$$

This condition determines only the translation part of \mathcal{M}_3 , and the remaining part is left arbitrary. For some of the specific vertices we consider later this condition is already met and so no conformal mapping is required on leg 3.

Having constructed $V^{[3]}(z)$, we can derive the vertex $\bar{V}^{[3]}(z, w, w)$ and work out the conformal mappings relating it to the initial $\check{V}^{[3]}$ as above. Let us take \mathcal{N}_j ($j = 1, 2, 3$) to be the conformal transformations between $V^{[3]}(z)$ and $\bar{V}^{[3]}(z, w, w)$, i.e.

$$\bar{V}^{[3]}(z, w, w) = V^{[3]}(z) \prod_j \hat{\mathcal{N}}_j^{(j)}. \quad (4.40)$$

From Eqs. (3.28), (4.34) and (4.35) we deduce the relations

$$\Gamma = (\mathcal{N}_1)^{-1} \circ \Gamma \circ \mathcal{N}_2 \quad (4.41)$$

$$\Gamma \circ \bar{\tau}_{31}(w; -) \circ \bar{\tau}_{13}(z; -) = (\mathcal{N}_1)^{-1} \circ \tau_{13}(z; -) \circ \mathcal{N}_3. \quad (4.42)$$

We can choose $\mathcal{N}_3 \equiv \text{id}$ and then

$$\mathcal{N}_1 = \bar{\tau}_{13}(w; -) \circ \Gamma, \quad \mathcal{N}_2 = \Gamma \circ \bar{\tau}_{13}(w; -). \quad (4.43)$$

In view of Eq. (3.32) we see that $\hat{\mathcal{N}}_1^\dagger$ corresponds to $(\mathcal{N}_2)^{-1}$.

From the general formula

$$\mathcal{R}_1 \left[V^{[3]}(z) \hat{\mathcal{N}}_1^{(1)} \hat{\mathcal{N}}_2^{(2)} | \varphi \rangle_3 \right] = \hat{\mathcal{N}}_1^\dagger \mathcal{V}(\varphi, z) \hat{\mathcal{N}}_2, \quad (4.44)$$

we obtain the associated one-string vertex operator, using $\mathcal{N} \equiv \mathcal{N}_2$, $\hat{\mathcal{N}}_1^\dagger \cong \mathcal{N}^{-1}$,

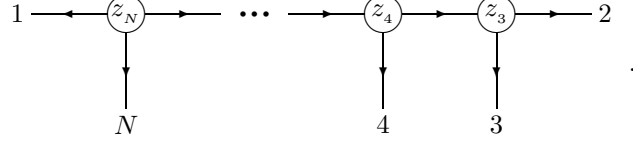
$$\begin{aligned} \bar{\mathcal{V}}(\varphi; z, w) &= \mathcal{R}_1 \left[V^{[3]}(z, w, w) | \varphi \rangle_3 \right] \\ &= \mathcal{R}_1 \left[V^{[3]}(z) \hat{\mathcal{N}}^{(1)} \hat{\mathcal{N}}^{(2)} | \varphi \rangle_3 \right] \\ &= \hat{\mathcal{N}}^\dagger \mathcal{V}(\varphi, z) \hat{\mathcal{N}} \\ &= \left[\frac{d\mathcal{N}^{-1}(z)}{dz} \right]^h \mathcal{V}(\varphi, \mathcal{N}^{-1}(z)). \end{aligned} \quad (4.45)$$

This relation tells us that when we put a state on their third leg then $V^{[3]}(z)$ and $\bar{V}^{[3]}(z, w, w)$ lead to one-string vertex operators that are related by the conformal transformation \mathcal{N}^{-1} . Note that it is the transformation \mathcal{N}^{-1} which feeds the w -dependence into the vertex operator $\bar{V}^{[3]}(z, w, w)$.

Sewing of vertices in string theory was traditionally performed by sewing vertices together with a propagator that involved a Feynman like parameter z (see e.g. [1]). In contrast, we have sewn the vertices $V^{[3]}(z)$ directly, i.e.

with no propagator factor. However, we can reconcile these two approaches, since the conformal transformation between $\check{V}^{[3]}$ and $V^{[3]}(z)$ can be interpreted as a propagator and in this way of calculating we sew vertices $\check{V}^{[3]}$ with propagator $\hat{\mathcal{M}}_2(z)\hat{\mathcal{M}}_1^\sharp(w)$ which, by Eqs. (3.32), (4.37) and (4.38), corresponds to the conformal transformation $\check{\tau}_{21} \circ \sigma_{\frac{z}{w}} \circ \Gamma$.

Finally, we can extend our calculation of the transition functions to the N -vertex



It is not difficult to arrive at the following results:

$$\begin{aligned}\tau_{12} &= \Gamma, \\ \tau_{1i} &= \tau_{13}(z_i) \quad \text{for } 3 \leq i \leq N,\end{aligned}$$

from which we deduce that

$$\begin{aligned}\tau_{i1} &= \tau_{31}(z_i) \quad \text{for } 3 \leq i \leq N, \\ \tau_{i2} &= \tau_{31}(z_i) \circ \Gamma \quad \text{for } 3 \leq i \leq N, \\ \tau_{ij} &= \tau_{31}(z_i) \circ \tau_{13}(z_j) \quad \text{for } 3 \leq i, j \leq N,\end{aligned}$$

expressed in terms of the transition function τ_{13} for the basic three-vertex. Thus we can write down a generalization of locality for the N -vertex:

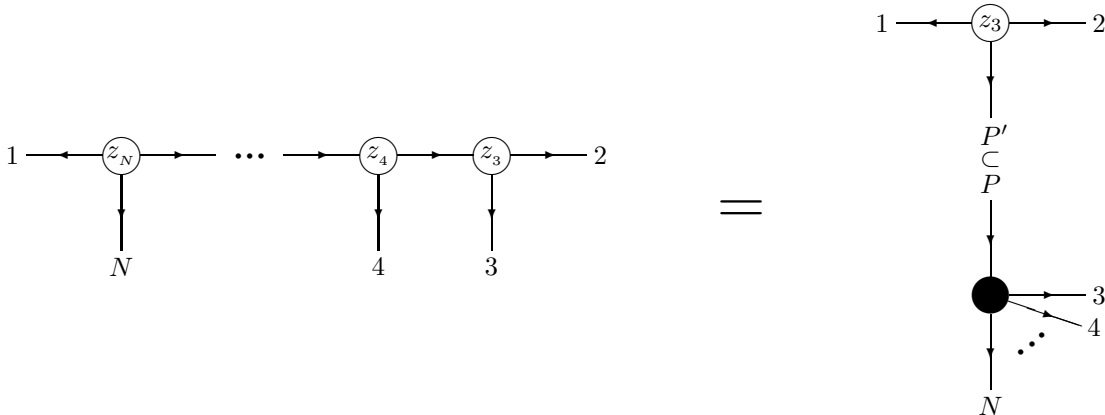
$$\tau_{ij}(z_i, z_j) = \tau_{ji}(z_j, z_i) \quad \text{for } 3 \leq i, j \leq N, \quad (4.46)$$

i.e. the N -vertex is symmetric (in the sense of analytic continuation of matrix elements) under simultaneous interchange of legs i and j and the variables z_i and z_j :

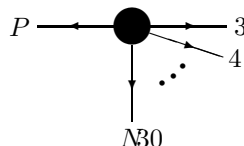
$$V_{1\dots N}^{[N]}(z_1, \dots, z_N) = V_{12k_3\dots k_N}^{[N]}(z_1, z_2, z_{k_3}, \dots, z_{k_N}) \quad (4.47)$$

for any permutation $\begin{pmatrix} 3 & \dots & N \\ k_3 & \dots & k_N \end{pmatrix}$.

We now wish to demonstrate an analogue of duality for the N -string vertex and construct the multistring analogue of $\check{V}^{[3]}(z, w, w)$. The sewing procedure is shown in the following figure:



The transition functions are calculated using the same techniques as before. Consequently, they must agree with the ones above, i.e. the vertex



has transition functions

$$\bar{\tau}_{P3} = \Gamma, \quad (4.48)$$

$$\bar{\tau}_{Pi} = \tau_{13}(z_i) \circ \tau_{31}(z_3) \circ \Gamma \quad \text{for } 4 \leq i \leq N, \quad (4.49)$$

$$\bar{\tau}_{ij} = \tau_{31}(z_i) \circ \tau_{13}(z_j) \quad \text{for } 3 \leq i, j \leq N. \quad (4.50)$$

4.3 Explicit construction of three-vertices

We will now demonstrate how easily the formalism works by studying two specific three-vertices, namely the CSV vertex V^{CSV} [9] and the special vertex $V^{[3]}(z)$ corresponding directly to the one-string vertices $\mathcal{V}(\psi, z)$ (for all choices of ψ) used in section 2. Studying different vertices and working out their properties is useful even though all vertices yield the same scattering amplitudes when sandwiched between physical states (when applying the formalism to hyperbolic Kac Moody algebras such as E_{10} , we will be concerned exclusively with physical states). The traditional example of a three-vertex is the *CSV*-vertex [9]; it is completely characterized by the overlap functions

$$\tau_{21} = \tau_{32} = \tau_{13} = \frac{1}{1-\xi}, \quad \tau_{31} = \tau_{23} = \tau_{12} = 1 - \frac{1}{\xi}, \quad (4.51)$$

and therefore manifestly cyclic invariant (i.e. $V_{123}^{\text{CSV}} = V_{231}^{\text{CSV}} = V_{312}^{\text{CSV}}$) with Koba Nielsen points $\infty, 0$ and 1 . For the matrices C_{mn}^{ij} occurring in (3.18), we obtain

$$\begin{aligned} C_{mn}^{12} &= C_{mn}^{23} = C_{mn}^{31} = (-1)^m \sqrt{\frac{m}{n}} \binom{n}{m}, \\ C_{mn}^{21} &= C_{mn}^{32} = C_{mn}^{13} = (-1)^n \sqrt{\frac{n}{m}} \binom{m}{n}, \end{aligned} \quad (4.52)$$

and

$$\begin{aligned} (\Gamma \circ \tau_{12})(0) &= (\Gamma \circ \tau_{23})(0) = (\Gamma \circ \tau_{31})(0) = 0, \\ (\Gamma \circ \tau_{21})(0) &= (\Gamma \circ \tau_{32})(0) = (\Gamma \circ \tau_{13})(0) = 1, \end{aligned} \quad (4.53)$$

so that the overlaps are given by

$$V^{\text{CSV}} \left\{ \alpha_{-m}^{(1)} + \sum_{n=1}^{\infty} (-1)^m \binom{n-1}{m-1} \alpha_n^{(2)} + \sum_{n=0}^m (-1)^n \binom{m}{n} \alpha_n^{(3)} \right\} = 0, \quad (4.54)$$

and cyclic permutations of this formula. With the above choice of transition functions the overlaps for the Virasoro generators become

$$V^{\text{CSV}} \left\{ L_{-m}^{(1)} - \sum_{n=0}^{\infty} (-1)^{m+n} \binom{1-m}{n} L_{m+n}^{(2)} + \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} L_{n-1}^{(3)} \right\} = 0, \quad (4.55)$$

again with their cyclically permuted analogues.

If we adopt V^{CSV} as our “initial” vertex, $\check{V}^{[3]}$, then the vertex $V^{[3]}(z)$ obtained by the procedure described in the foregoing section has the transition functions

$$\begin{aligned} \tau_{12} &= \Gamma, \\ \tau_{13} &= \sigma_{z^{-1}} \circ \check{\tau}_{13} = \frac{\xi - 1}{z\xi}, \\ \tau_{23} &= \Gamma \circ \sigma_{z^{-1}} \circ \check{\tau}_{13} = \frac{z\xi}{\xi - 1}, \end{aligned} \quad (4.56)$$

where we made use of Eqs. (3.28), (4.37) and (4.38). In the above we have taken $\mathcal{M}_3 = \text{id}$ as the condition $\mathcal{M}_3(0) = \check{\tau}_{31}(1) = 0$ is then satisfied. We can also find the corresponding vertex $\bar{V}^{[3]}(z, w, w)$ which has the

transition functions

$$\begin{aligned}
\bar{\tau}_{12} &= \Gamma, \\
\bar{\tau}_{31} &= \check{\tau}_{31} \circ \sigma_{\frac{z}{w}} \circ \check{\tau}_{13} \circ \Gamma, \\
\bar{\tau}_{32} &= \check{\tau}_{31} \circ \sigma_{\frac{z}{w}} \circ \check{\tau}_{13},
\end{aligned} \tag{4.57}$$

by Eq. (4.30).

As our second example we would like to identify the particular three-vertex which makes complete contact with the one-string vertex operators of Sect. 2, i.e. that three-vertex $V^{[3]}(z)$ whose associated one-string vertex operators $\mathcal{V}(\psi, z)$ (defined by means of (4.3)) satisfy *all* the axioms of Sect. 2 and not just those required in Sect. 4.1. For this special vertex and the choice $y = w$ the vertex $\bar{V}_{6'53}$ will be the same as the original one with the new argument $z - w$; this is just duality (cf. Theorem 2). Let us remind the reader that although the particular three-vertex we are about to construct has especially simple properties and permits us to recover all the properties given in Sect. 2, it is just one of an infinite number of three-vertices that are equivalent once we sandwich them or their associated N -vertices against physical states.

As before we need only work out the transition functions τ_{12}, τ_{23} and $\tau_{31} = (\tau_{13})^{-1}$, from which the associated three-vertex follows directly as explained in the preceding section. The task can thus be reduced to translating the following relations (see (2.8), (2.12), (2.13))

$$\begin{aligned}
\frac{d}{dz} \mathcal{V}(\varphi, z) &= \mathcal{V}(L_{-1}\varphi, z), \\
[L_0, \mathcal{V}(\varphi, z)] &= \left(z \frac{d}{dz} + h \right) \mathcal{V}(\varphi, z), \\
[L_{-1}, \mathcal{V}(\varphi, z)] &= \frac{d}{dz} \mathcal{V}(\varphi, z),
\end{aligned} \tag{4.58}$$

into the corresponding ones for the associated three-vertex $V^{[3]}$, which read

$$\begin{aligned}
\frac{d}{dz} V^{[3]}(z) - V^{[3]}(z) L_{-1}^{(3)} &= 0, \\
V^{[3]}(z) \left\{ -L_0^{(2)} + L_0^{(1)} - L_0^{(3)} - z L_{-1}^{(3)} \right\} &= 0, \\
V^{[3]}(z) \left\{ -L_{-1}^{(2)} + L_{-1}^{(1)} - L_{-1}^{(3)} \right\} &= 0,
\end{aligned} \tag{4.59}$$

and matching them with the overlap conditions for the Virasoro generators. It is important that in contrast to the assumptions previously made we here do not require the state φ to be physical; otherwise all equations would only hold up to terms involving $L_m^{(3)}$ for $m \geq 1$ which would remain undetermined. Demanding the vertices to agree off-shell is clearly a much more stringent requirement, and it is therefore not surprising that we will arrive at a unique answer for $V^{[3]}(z)$ in this case. It is also remarkable that the conditions for $m = -1, 0$ are sufficient to fix all the higher order identities.

Equation (4.59) can now be matched with (3.23) for $f(\xi) = \xi$ and $f(\xi) = \xi^2$, respectively. In this way, we deduce the conditions

$$\begin{aligned}
\xi_1 \frac{d\xi_2}{d\xi_1} &= -\xi_2, & \xi_1 \frac{d\xi_3}{d\xi_1} &= -\xi_3 - z, \\
(\xi_1)^2 \frac{d\xi_2}{d\xi_1} &= -1, & (\xi_1)^2 \frac{d\xi_3}{d\xi_1} &= -1.
\end{aligned} \tag{4.60}$$

These differential equations are solved by $\xi_i = \tau_{ij}(\xi_j)$ with

$$\begin{aligned}
\tau_{12}(\xi) &= \tau_{21}(\xi) = \frac{1}{\xi} \equiv \Gamma(\xi), \\
\tau_{13}(\xi) &= \frac{1}{\xi + z} \iff \tau_{31}(\xi) = \frac{1}{\xi} - z, \\
\tau_{23}(\xi) &= \xi + z.
\end{aligned} \tag{4.61}$$

Identifying the coordinate ξ_2 with the global coordinate ζ , we get $z_2 = 0$ for the second Koba Nielsen point; the other two follow from $z_i = \tau_{2i}(0)$ and come out to be $z_1 = \infty$ and $z_3 = z$.

As an illustration let us explicitly derive the tachyon emission vertex operator (cf. (2.74)). We put the tachyonic state $|\mathbf{v}\rangle$ on the third leg. Consequently, in formula (3.18) all terms involving annihilation operators $\alpha_n^{(3)}$ vanish, viz.

$$V^{[3]}(z)|\mathbf{v}\rangle_3 = \langle \tilde{0} | \exp \left\{ - \sum_{m,n \geq 1} \frac{\alpha_m^{(1)}}{\sqrt{m}} C_{mn}^{12} \frac{\alpha_n^{(2)}}{\sqrt{n}} - \sum_{i=1,2} \sum_{m \geq 1} \frac{\mathbf{v} \cdot \alpha_m^{(i)}}{m} [(\Gamma \circ \tau_{i3})(0)]^m \right\} \mathcal{N}(\{\alpha_0^{(i)}\}) |\mathbf{v}\rangle_3,$$

where we have also taken into account that $(\Gamma \circ \tau_{12})(0) = (\Gamma \circ \tau_{21})(0) = 0$ for our choice of τ_{12} . If we furthermore insert $(\Gamma \circ \tau_{13})(0) = z$, $(\Gamma \circ \tau_{23})(0) = z^{-1}$, $C_{mn}^{12} = \delta_{mn}$ and turn around the first leg we get

$$\mathcal{R}_1 V^{[3]}(z)|\mathbf{v}\rangle_3 = \sum_{\substack{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \Lambda \\ \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0}} {}_2\langle \mathbf{r}_2 | {}_3\langle \mathbf{r}_3 | \exp \left\{ \sum_{m \geq 1} \frac{1}{m} \alpha_{-m}^{(1)} \cdot \alpha_m^{(2)} + i\mathbf{v} \cdot \mathbf{X}_{<}^{(1)}(z) + i\mathbf{v} \cdot \mathbf{X}_{>}^{(2)}(z) \right\} z^{\mathbf{v} \cdot \mathbf{r}_2} | -\mathbf{r}_1 \rangle_1 |\mathbf{v}\rangle_3,$$

also invoking formula (3.13) for the zero mode term \mathcal{N} (and discarding some phase factor). We can now use $\langle \mathbf{r}_3 | \mathbf{v} \rangle = \delta_{\mathbf{v}, \mathbf{r}_3}$ and perform the summation over \mathbf{r}_1 and \mathbf{r}_3 to arrive at

$$\mathcal{R}_1 V^{[3]}(z)|\mathbf{v}\rangle_3 = \sum_{\mathbf{r}_2 \in \Lambda} {}_2\langle \mathbf{r}_2 | \exp \left\{ \sum_{m \geq 1} \frac{1}{m} \alpha_{-m}^{(1)} \cdot \alpha_m^{(2)} + i\mathbf{v} \cdot \mathbf{X}_{<}^{(1)}(z) + i\mathbf{v} \cdot \mathbf{X}_{>}^{(2)}(z) \right\} z^{\mathbf{v} \cdot \mathbf{r}_2} | \mathbf{r}_2 \rangle_1.$$

Finally, we consider the action of this expression on an arbitrary state ψ on leg 2. The role of the bilinear term in the exponent is easily exhibited by noticing that it is nothing but the natural isomorphism Π_{12} (cf. (4.15)) between the Fock spaces \mathcal{F}_2 and \mathcal{F}_1 :

$$\Pi_{12} := \mathcal{R}_1 V^{[3]}(z)|0\rangle_3 = \sum_{\mathbf{r}_2 \in \Lambda} {}_2\langle \mathbf{r}_2 | \exp \left(\sum_{m \geq 1} \frac{1}{m} \alpha_{-m}^{(1)} \cdot \alpha_m^{(2)} \right) | \mathbf{r}_2 \rangle_1. \quad (4.62)$$

One can explicitly verify that

$$\Pi_{12} \alpha_{n_1}^{(2)} \cdots \alpha_{n_N}^{(2)} | \mathbf{r} \rangle_2 = \alpha_{n_1}^{(1)} \cdots \alpha_{n_N}^{(1)} | \mathbf{r} \rangle_1. \quad (4.63)$$

Then, by linearity, one concludes that the operator Π_{12} indeed identifies \mathcal{F}_2 with \mathcal{F}_1 . The inverse of Π_{12} is obviously given by Π_{21} and we also have ${}_1\langle \psi | \Pi_{12} = {}_2\langle \psi |$. The above expression we are interested in can now be written as

$$e^{i\mathbf{v} \cdot \mathbf{X}_{<}^{(1)}(z)} e^{i\mathbf{v} \cdot \mathbf{q}^{(1)}} c_{\mathbf{v}}^{(1)} z^{\mathbf{v} \cdot \alpha_0^{(1)}} \Pi_{12} e^{i\mathbf{v} \cdot \mathbf{X}_{>}^{(2)}(z)}, \quad (4.64)$$

where we have extracted from $|\mathbf{r}_2 + \mathbf{v}\rangle_1$ the factor $e_{\mathbf{v}} = e^{i\mathbf{v} \cdot \mathbf{q}_{\mathbf{v}}}$. Using the properties of Π_{12} we infer that we indeed recover the desired formula (2.74). Note that the map Π_{12} is also responsible for the correct normal-ordering of the final vertex operator.

The reader may be curious about the conformal transformations \mathcal{M}_j which relate the above three-vertex to the CSV vertex, i.e.

$$V^{[3]} = V^{\text{CSV}} \hat{\mathcal{M}}_1 \hat{\mathcal{M}}_2 \hat{\mathcal{M}}_3. \quad (4.65)$$

Applying the general prescription explained after Eq. (4.37) we arrive at

$$\mathcal{M}_1(\xi) = z\xi, \quad \mathcal{M}_2(\xi) = \frac{\xi}{\xi - z}, \quad \mathcal{M}_3(\xi) = -z^{-1}\xi, \quad (4.66)$$

or, as operators,

$$\hat{\mathcal{M}}_1 = z^{L_0}, \quad \hat{\mathcal{M}}_2 = z^{-L_0} (-1)^{L_0} e^{z^{-1}L_1}, \quad \hat{\mathcal{M}}_3 = z^{-L_0} (-1)^{L_0}. \quad (4.67)$$

Thus, sewing together $V^{[3]}(z)$ and $V^{[3]}(w)$ is equivalent to sewing two V^{CSV} 's with propagator

$$\hat{\mathcal{M}}_2(z) \hat{\mathcal{M}}_1^\#(w) = z^{-L_0} (-1)^{L_0} e^{z^{-1}L_1} w^{L_0} = (-1)^{L_0} e^{L_1} \left(\frac{w}{z} \right)^{L_0}. \quad (4.68)$$

The first two factors on the right-hand side combine into what was called twist-factor in the old days (see e.g. [1]).

If we insert our special choice of transition functions into the general formulas (4.31)–(4.33) for the transition functions of the four-vertex we find

$$\bar{\tau}_{56'}(\xi_{6'}) = \frac{1}{\xi_{6'}} + y - w, \quad (4.69)$$

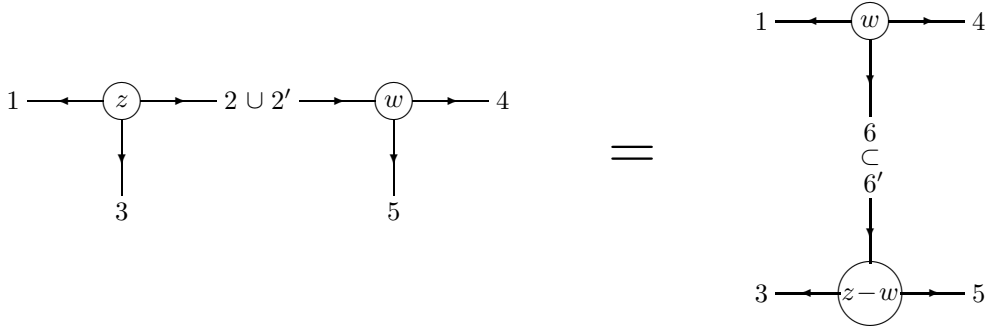
$$\bar{\tau}_{36'}(\xi_{6'}) = \frac{1}{\xi_{6'}} + y - z, \quad (4.70)$$

$$\bar{\tau}_{53}(\xi_3) \equiv \tau_{53}(\xi_3) = \xi_3 + z - w. \quad (4.71)$$

Setting $y = w$, we finally obtain

$$\begin{aligned} \bar{\tau}_{56'}(\xi_{6'}) &= \frac{1}{\xi_{6'}} \quad \text{i.e.} \quad \bar{\tau}_{6'5} = \Gamma = \tau_{12}, \\ \bar{\tau}_{36'}(\xi_{6'}) &= \frac{1}{\xi_{6'}} + w - z \quad \text{i.e.} \quad \bar{\tau}_{6'3} = \tau_{13}(z - w; -), \\ \bar{\tau}_{53}(\xi_3) &= \xi_3 + z - w \quad \text{i.e.} \quad \bar{\tau}_{53} = \tau_{23}(z - w; -), \end{aligned} \quad (4.72)$$

so that indeed $\bar{V}_{6'53}(z, w, w) \equiv V_{6'53}(z - w)$ and we have established duality, viz.



The symmetry of the new vertex under simultaneous interchanges of z and w and legs 3 and 5 is now manifest because $\tau_{53}(z - w) = \tau_{35}(w - z)$. The above identity is nothing but (2.10) for vertex algebras, but now expressed in terms of three-vertices. In view of Theorem 2 we have therefore established the Jacobi identity (2.10) for the vertex operators defined by Eq. (4.3). Note that in order to get the required off-shell behaviour, we had to fix all transition functions. We hope that readers will appreciate the simplicity of our proof of this identity which is substantially shorter, involving only simple manipulations with overlaps and transition functions, and at the same time more easily visualized than the one which can be found in the textbook [21].

The transition functions for the N -vertex are given by

$$\begin{aligned} \tau_{12} &= \Gamma, \\ \tau_{1i} &= \frac{1}{\xi_i + z_i} \quad \text{for } 3 \leq i \leq N. \end{aligned}$$

If we put formally $z_2 \equiv 0$ then these relations can be elegantly summarized as

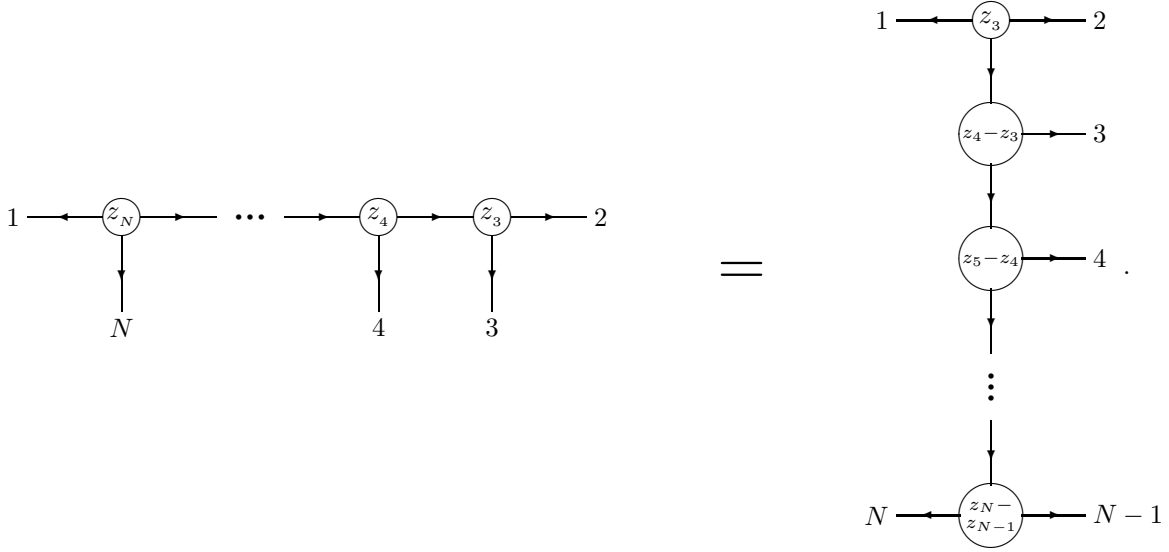
$$\tau_{1i} = \frac{1}{\xi_i + z_i} \quad \text{for } 2 \leq i \leq N,$$

from which we deduce that

$$\begin{aligned} \tau_{i1} &= \frac{1}{\xi_1} - z_i \quad \text{for } 2 \leq i \leq N, \\ \tau_{ij} &= \tau_{i1} \circ \tau_{1j} = \xi_j + z_j - z_i \quad \text{for } 2 \leq i, j \leq N. \end{aligned}$$

Recall that it was this choice of transition functions which led to the especially simple result $\mu = 1$ for the measure. Identifying the coordinate ξ_2 with the global coordinate ζ , we find that $\tau_{2j}(0) = z_j$ for $2 \leq j \leq N$ so that the Koba Nielsen points indeed agree with the above formal variables z_j for $2 \leq j \leq N$; whereas for z_1 we obtain the previous result ∞ . The symmetry of the N -vertex under simultaneous interchange of legs i and j and the variables z_i and z_j is now explicit: $\tau_{ij}(z_i - z_j) = \tau_{ji}(z_j - z_i)$ for $2 \leq i, j \leq N$.

We may also repeatedly apply the duality property to the N -vertex to rewrite it as follows:



Now we can write down the overlap equations for the oscillators for this particular choice of $V^{[N]}$, as we will need them in Sect. 5. We find

$$\begin{aligned} C_{mn}^{12} &= \delta_{mn}, & (\Gamma \circ \tau_{12})(0) &= 0, \\ C_{mn}^{1i} &= \sqrt{\frac{n}{m}} \binom{m}{n} z_i^{m-n}, & (\Gamma \circ \tau_{1i})(0) &= z_i \quad \text{for } 3 \leq i \leq N. \end{aligned} \quad (4.73)$$

Thus

$$V^{[N]}(z_3, \dots, z_N) \left\{ \alpha_{-m}^{(1)} + \alpha_m^{(2)} + \sum_{i=3}^N \sum_{n=0}^m \binom{m}{n} z_i^{m-n} \alpha_n^{(i)} \right\} = 0. \quad (4.74)$$

The Virasoro overlaps are given by

$$V^{[N]}(z_3, \dots, z_N) \left\{ L_{-m}^{(1)} - L_m^{(2)} - \sum_{j=3}^N \sum_{n=-1}^m \binom{m+1}{m-n} z_j^{m-n} L_n^{(j)} \right\} = 0. \quad (4.75)$$

We also record the generalization of (4.59) for arbitrary N -vertices:

$$\begin{aligned} \frac{\partial}{\partial z_j} V^{[N]}(z_3, \dots, z_N) - V^{[N]}(z_3, \dots, z_N) L_{-1}^{(j)} &= 0 \quad (j \geq 3), \\ V^{[N]}(z_3, \dots, z_N) \left\{ L_0^{(1)} - \sum_{j=2}^N (L_0^{(j)} + z_j L_{-1}^{(j)}) \right\} &= 0, \\ V^{[N]}(z_3, \dots, z_N) \left\{ L_1^{(1)} - \sum_{j=2}^N L_{-1}^{(j)} \right\} &= 0, \end{aligned} \quad (4.76)$$

where, of course, $z_2 = 0$. With the help of the above overlap equations we can now explicitly verify that null states decouple when all legs of the vertex except leg 1 are saturated with physical states ψ_2, \dots, ψ_N . A short calculation shows that

$$V^{[N]}(z_3, \dots, z_N) |\psi_2\rangle_2 \dots |\psi_N\rangle_N L_{-m}^{(1)} = \sum_{j=3}^N \frac{\partial}{\partial z_j} \left(z_j^{m+1} V^{[N]}(z_3, \dots, z_N) \right) |\psi_2\rangle_2 \dots |\psi_N\rangle_N, \quad (4.77)$$

which vanishes upon integration over the variables z_3, \dots, z_N . Note that this relation is stronger than (4.4) since leg 1 is not contracted with a state.

We now come back to our claim at the beginning of this section that the three-vertex may be regarded as an intertwining operator. To see this let us recall the definition of such operators as given in [19] but with suitably modified notation.

Definition 4

Let $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$ be a vertex algebra and let $(\mathcal{F}_i, \mathcal{V}^{(i)})$, $i = 1, 2, 3$, be three modules for the vertex algebra. An **intertwining operator of type** $\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_3 \mathcal{F}_2 \end{pmatrix}$ is a linear map $\mathcal{I} : \mathcal{F}_3 \rightarrow (\text{Hom}(\mathcal{F}_2, \mathcal{F}_1))\{z\}$, $|\psi\rangle_3 \mapsto \mathcal{I}(z)|\psi\rangle_3$, satisfying the following properties: For every $|\xi\rangle_1 \in \mathcal{F}_1$, $|\varphi\rangle_2 \in \mathcal{F}_2$, $|\psi\rangle_3 \in \mathcal{F}_3$,

1. **(Regularity)**

$$\text{Res}_z [z^n \mathcal{I}(z)|\psi\rangle_3 |\varphi\rangle_2] = 0 \quad \text{for } n \text{ sufficiently large;} \quad (4.78)$$

2. **(Translation)**

$$\frac{d}{dz} \mathcal{I}(z)|\psi\rangle_3 = \mathcal{I}(z)L_{-1}^{(3)}|\psi\rangle_3; \quad (4.79)$$

3. **(Jacobi identity)**

$$\begin{aligned} y^{-1} \delta \left(\frac{z-w}{y} \right) \mathcal{V}^{(1)}(\xi, z) \mathcal{I}(w)|\psi\rangle_3 |\varphi\rangle_2 - y^{-1} \delta \left(\frac{-w+z}{y} \right) \mathcal{I}(w)|\psi\rangle_3 \mathcal{V}^{(2)}(\xi, z)|\varphi\rangle_2 \\ = w^{-1} \delta \left(\frac{z-y}{w} \right) \mathcal{I}(w) \mathcal{V}^{(3)}(\xi, y)|\psi\rangle_3 |\varphi\rangle_2, \end{aligned} \quad (4.80)$$

where binomial expressions have to be expanded in nonnegative integral powers of the second variable.

We are here interested in the case where $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are isomorphic copies of \mathcal{F} and represent modules for the vertex algebra via the adjoint action. Consequently, the intertwiner \mathcal{I} leads to the rather trivial fusion rules $N_{32}^1 = 1$. The interesting point, however, is that the overlap equations for $V^{[3]}$ are very reminiscent to the Jacobi identity for the intertwining operator \mathcal{I} . In fact, we can show that the two properties are equivalent to each other in the same way as the Jacobi identity for vertex operators (2.10) is related to the principles of locality and duality in Theorem 1. To see this, we take matrix elements of the Jacobi identity for interwiners, i.e. we pair (4.80) with an arbitrary state ${}_1\langle\chi|$ from the left, and then apply $\text{Res}_y [\dots]$. Using the identity

$$w^{-1} \delta \left(\frac{z-y}{w} \right) = y^{-1} \delta \left(\frac{z-w}{y} \right) - y^{-1} \delta \left(\frac{-w+z}{y} \right), \quad (4.81)$$

we can mimic the proof of Theorem 1 as presented in [19] and finally arrive at the following result.

Theorem 4

1. For $\chi, \psi, \varphi, \xi \in \mathcal{F}$, the formal series ${}_1\langle\chi|\mathcal{V}^{(1)}(\xi, z)\mathcal{I}(w)|\psi\rangle_3 |\varphi\rangle_2$ which involves only finitely many negative powers of w and only finitely many positive powers of z , lies in the image of the map ι_{zw} :

$${}_1\langle\chi|\mathcal{V}^{(1)}(\xi, z)\mathcal{I}(w)|\psi\rangle_3 |\varphi\rangle_2 = \iota_{zw} f(z, w), \quad (4.82)$$

where the (uniquely determined) element $f \in \mathbf{C}[z, w]_S$ is of the form

$$f(z, w) = \frac{g(z, w)}{z^r w^s (z-w)^t}$$

for some polynomial $g(z, w) \in \mathbf{C}[z, w]$ and $r, s, t \in \mathbf{Z}$. We also have

$${}_1\langle\chi|\mathcal{I}(w)|\psi\rangle_3 \mathcal{V}^{(2)}(\xi, z)|\varphi\rangle_2 = \iota_{wz} f(z, w), \quad (4.83)$$

i.e. $\mathcal{V}^{(1)}(\xi, z)\mathcal{I}(w)|\psi\rangle_3$ agrees with $\mathcal{I}(w)|\psi\rangle_3 \mathcal{V}^{(2)}(\xi, z)$ as operator-valued rational functions.

2. For $\chi, \psi, \varphi, \xi \in \mathcal{F}$, the formal series ${}_1\langle\chi|\mathcal{I}(w)\mathcal{V}^{(3)}(\xi, y)|\psi\rangle_3 |\varphi\rangle_2$ which involves only finitely many negative powers of y and only finitely many positive powers of w , lies in the image of the map ι_{wy} :

$${}_1\langle\chi|\mathcal{I}(w)\mathcal{V}^{(3)}(\xi, y)|\psi\rangle_3 |\varphi\rangle_2 = \iota_{wy} f(y + w, w), \quad (4.84)$$

with the same f as above, and

$${}_1\langle\chi|\mathcal{V}^{(1)}(\xi, y+w)\mathcal{I}(w)|\psi\rangle_3 |\varphi\rangle_2 = \iota_{yw} f(y + w, w), \quad (4.85)$$

i.e. $\mathcal{V}^{(1)}(\xi, z)\mathcal{I}(w)|\psi\rangle_3$ agrees with $\mathcal{I}(w)\mathcal{V}^{(3)}(\xi, z-w)|\psi\rangle_3$ as operator-valued rational functions, where the right-hand expression is to be expanded as a Laurent series in $z-w$.

If we now put $\mathcal{I}(z) \equiv \mathcal{R}_1 V^{[3]}(z)$ then the above theorem is just a rigorous statement about the unintegrated overlap identities for the three-vertex $V^{[3]}(z)$ with transition functions (4.61). We note, however, that the intertwiner Jacobi identity refers to a special choice of transition functions whereas the unintegrated overlap equations are valid in general. Hence it is reasonable to expect that there is an extension of (4.80) which does not require a specification of the transition functions. This would be the rigorous (in the sense of formal calculus) analog of the overlap identities. Besides from that point, the overlap identities are apparently much more profound than the Jacobi identity for intertwiners since they also hold for N -vertices. Finally, we emphasize that (4.80) is stated as a *postulate* for the three-vertex in the mathematical literature whereas it is proved here on the basis of an explicit realization.

5 N -string vertices and E_{10}

We now come to the heart of this paper, applying the multistring formalism developed in the foregoing sections to the study of hyperbolic Kac Moody algebras. For definiteness, we will only consider E_{10} in the remainder, although some of our results are actually more general, especially those concerning longitudinal states. Recall that for a given Cartan matrix A the associated Kac Moody algebra $\mathfrak{g}(A)$ is recursively defined in terms of multiple commutators of the basic Chevalley generators as explained in Sect. 2 after (2.88). Hyperbolic Cartan matrices are indefinite and constrained by the requirement that the deletion of any node from the Dynkin diagram leaves a diagram which is either of finite or affine type. It can then be shown that they possess one and only one negative eigenvalue; the corresponding “over-extended” root will always be denoted as \mathbf{r}_{-1} (the affine root is denoted by \mathbf{r}_0). Furthermore, the root lattice is Minkowskian in this case. By an important result of [7] the rank of a hyperbolic Kac Moody algebra cannot exceed $d = 10$. Among the rank $d = 10$ algebras the most interesting is E_{10} with the even self-dual Lorentzian lattice $\Lambda = \mathbb{I}_{9,1}$ as its root lattice. The central unsolved problem in the theory of hyperbolic Kac Moody algebras is to find a manageable representation for the multiple commutators of the Chevalley generators analogous to the explicit realizations of the finite and affine algebras; specifically, this requires to identify and discard all those multiple commutators containing the Serre relations (2.88) somewhere inside.

From Sect. 2 we know already that for any given lattice Λ (and in particular, any root lattice) one can construct a Lie algebra of physical string states denoted by \mathfrak{g}_Λ . For indefinite lattices, this Lie algebra, which is relatively easy to characterize, is *not* the same as the Kac Moody algebra $\mathfrak{g}(A)$ generated by the Chevalley states (2.84), but contains it as a *proper* subalgebra according to (2.90). The problem then becomes one of identifying the decoupled states which are in \mathfrak{g}_Λ but not in $\mathfrak{g}(A)$, i.e. those physical states which cannot be reached via multiple commutators. The multistring formalism developed in the foregoing sections furnishes a new method to analyze and compute multiple commutators. This is achieved by rewriting any N -fold multiple commutator in terms of an $(N + 2)$ -string vertex by attaching the Lie algebra elements appearing in the commutator to the legs of this vertex. This method permits an efficient and reasonably “easy” determination of the decoupled states by means of overlap identities. Thus, instead of constructing the root space elements directly we will search for the physical states which are in the complement of the root space. An essential tool for this purpose is the DDF construction, suitably adapted to the present context. For “subcritical” Kac Moody algebras such as E_{10} , both transversal and longitudinal DDF operators can be shown to appear. To demonstrate the utility of the new approach, we will show how to recover the results of Sect. 4.4 of [26] by explicitly determining which states in the root space $E_{10}^{(A_7)}$ decouple from the three-vertex. As for the longitudinal states, our results go considerably beyond those of [26] and quite some way towards a more systematic understanding. The decoupling of transversal states is even more remarkable, as it depends on the lattice in a crucial manner and has no analog for continuous momenta unlike the longitudinal decoupling; for the time being it remains a mysterious phenomenon. Nonetheless, we hope that the results presented here convincingly support our claim that the entire information about hyperbolic algebras such as E_{10} is encapsulated in the N -vertices of the compactified string!

5.1 The Lie algebra of physical states in terms of N -vertices

Before turning to the hyperbolic Kac Moody algebra E_{10} , we consider the Lie algebra of physical states \mathfrak{g}_Λ discussed in detail in Sect. 2. We recall that the commutator of two physical states is given by Eq. (2.38). Using (4.2) we can reexpress it by means of the special three-vertex $V^{[3]}(z)$ corresponding to the transition functions (4.61). Specifically, the commutator of two physical states ψ, φ in the Lie algebra of physical states \mathfrak{g}_Λ is

$$[\psi, \varphi] = \text{Res}_z [\mathcal{V}(\psi, z)\varphi]$$

$$= \text{Res}_z \left\{ \mathcal{R}_1 \left[V_{123}^{[3]}(z) |\varphi\rangle_2 |\psi\rangle_3 \right] \right\}. \quad (5.1)$$

We use the following pictorial representation of the commutator:

$$[\psi, \varphi] \longleftrightarrow \text{Res}_z \left[\begin{array}{c} \longrightarrow \textcircled{z} \longrightarrow |\varphi\rangle_2 \\ \downarrow \\ |\psi\rangle_3 \end{array} \right]$$

We have thus a new and convenient way to calculate the commutator by contracting two of the three legs of $V^{[3]}$ with the physical states φ and ψ . Although the state $|\tilde{0}\rangle$ is not normalizable, the N -vertex does give rise to a normalizable Fock space state upon contraction with $N - 1$ states. The resulting element of \mathfrak{g}_Λ can be visualized as the final state of a scattering process of the states φ and ψ . As already pointed out before (cf. Eq. (2.42)), the result makes sense as a Lie algebra commutator only on the factor space $\mathcal{P}^1/L_{-1}\mathcal{P}^0$. This means that the final state should be orthogonal to all states of the form $L_{-1}\chi$ with $\chi \in \mathcal{P}^0$; this decoupling is, however, a consequence of (4.4). For more general N -vertices, it can be established by means of Virasoro overlaps along the lines of (4.77).

We next work out some simple consequences of Eq. (5.1). Let $\psi, \varphi, \chi \in \mathfrak{g}_\Lambda$. Then

$$\begin{aligned} \text{Res}_z \left[V^{[3]}(z) |\chi\rangle_1 |\varphi\rangle_2 |\psi\rangle_3 \right] &= (\chi | [\psi, \varphi]) \\ &= (\varphi | [\chi, \psi]) \\ &= \text{Res}_z \left[V^{[3]}(z) |\varphi\rangle_1 |\psi\rangle_2 |\chi\rangle_3 \right], \end{aligned} \quad (5.2)$$

Consequently, invariance and symmetry of the bilinear form $(-|-)$ are equivalent to the on-shell cyclicity of the three-vertex $V^{[3]}$. Note that although only the CSV vertex is manifestly cyclic, upon contraction with physical states all three-vertices are cyclic.

The structure constants of \mathfrak{g}_Λ are defined by

$$\begin{aligned} f_{\psi\varphi}^\chi &:= \langle [\psi, \varphi] | \chi \rangle \\ &= -(\theta(\chi) | [\psi, \varphi]) \\ &= -\text{Res}_z \left[V^{[3]}(z) |\theta(\chi)\rangle_1 |\varphi\rangle_2 |\psi\rangle_3 \right] \\ &= {}_1\langle \chi | \text{Res}_z \left[\mathcal{R}_1 V^{[3]}(z) \right] |\varphi\rangle_2 |\psi\rangle_3. \end{aligned} \quad (5.3)$$

Note that the vertex with one leg turned around gives the usual string scalar product whereas the bare vertex leads to the invariant form for \mathfrak{g}_Λ . We can “pull down” the index χ by writing

$$f_{\psi\varphi\chi} := (\chi | [\psi, \varphi]). \quad (5.4)$$

The corresponding metric is just the Cartan Killing metric on \mathfrak{g}_Λ . Since all the states used in (5.3) are physical and therefore inert under the conformal transformations (3.26), the structure constants (5.3) are insensitive to the off-shell properties of $V^{[3]}$.

Pictorially, we have

$$f_{|3\rangle|2\rangle}^{[1]} \longleftrightarrow \text{Res}_z \left[\begin{array}{c} \langle 1 | \longrightarrow \textcircled{z} \longrightarrow | 2 \rangle \\ \downarrow \\ | 3 \rangle \end{array} \right].$$

Evidently all the information about the structure constants and hence the Lie algebra \mathfrak{g}_Λ is encoded into the vertex $\text{Res}_z [\mathcal{R}_1 V^{[3]}(z)]$ in this manner. Note that in the above formulas for the structure constants we may drop the residue and put $z_3 \equiv z = 1$ since we still have the freedom to fix a third (besides $z_1 = \infty$ and $z_2 = 0$) Koba Nielsen point due to global Möbius invariance.

Next we iterate (5.1) to obtain a formula for the multiple commutator. Evaluating the product of three-vertices amounts to sewing them as we explained, which yields the formula

$$[\psi_N, [\psi_{N-1}, \dots, [\psi_3, \psi_2] \dots]] = \text{Res}_{z_N} \dots \text{Res}_{z_4} \text{Res}_{z_3} \left[\mathcal{R}_1 V^{[N]}(z_3, \dots, z_N) \right] |\psi_2\rangle_2 \dots |\psi_N\rangle_N \quad (5.5)$$

generalizing (5.1) and (2.38). It is then natural to extend the above definition (5.3) and to introduce structure constants for $(N-2)$ -fold commutators for $N \geq 4$. These are perhaps less familiar objects but nonetheless very convenient. More specifically, the generalized structure constants are associated with $\text{Res}_{z_N \dots z_4 z_3} [\mathcal{R}_1 V^{[N]}(\{z_i\})]$, where z_3, \dots, z_N denote the formal variables with respect to which the residues are to be computed:

$$\begin{aligned} f_{\psi_N \dots \psi_2}^{\psi_1} &:= \langle [\psi_N, [\psi_{N-1}, \dots [\psi_3, \psi_2]] \dots] | \psi_1 \rangle \\ &= {}_1 \langle \psi_1 | \text{Res}_{z_N} \dots \text{Res}_{z_4} \text{Res}_{z_3} \left[\mathcal{R}_1 V^{[N]}(z_3, z_4, \dots, z_N) \right] | \psi_2 \rangle_2 \dots | \psi_N \rangle_N. \end{aligned} \quad (5.6)$$

Symbolically,

$$f_{|N\rangle \dots |2\rangle}^{(1)} \longleftrightarrow \text{Res}_{z_N, \dots, z_4, z_3} \left[\begin{array}{c} \langle 1 | \longrightarrow \textcircled{z_N} \longrightarrow \dots \longrightarrow \textcircled{z_4} \longrightarrow \textcircled{z_3} \longrightarrow | 2 \rangle \\ \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \\ | N \rangle \qquad \qquad \qquad | 4 \rangle \qquad \qquad | 3 \rangle \end{array} \right].$$

We stress that the special vertex characterized by (4.61) is just such that, for any N -vertex obtained from it by sewing, the relevant factors in (3.1) combine to give unity upon fixing the first three Koba Nielsen points to ∞ , 0 and 1, respectively, so we need no longer worry about them. In other words, the generalized structure constants for an $(N-2)$ -fold commutator of physical string states are obtained by simply integrating the saturated N -vertex over the Koba Nielsen variables z_4, \dots, z_N without any extra factors. We have already pointed out that since all attached states are physical, we would obtain the same result with any other N -vertex; however, we then would have to keep track of the extra measure factors in (3.1), which would no longer equal unity.

For completeness and to establish the link with the more traditional vertex operator construction based on (2.44), we rephrase the above results in terms of (ordinary) commutators of multistring vertices. We stress that the latter are a priori different from the prescription (2.38), which maps two physical states to another physical state in a way compatible with the properties of a Lie bracket. As already mentioned the integrated vertex operators defined via (4.3) realize the adjoint representation of the Lie algebra of physical states. Carrying out the commutation of two such operators corresponds to sewing $\mathcal{R}_1 V_{123}(z) |\psi\rangle_3 \overset{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) |\varphi\rangle_5$ and integrating their antisymmetrized combination with respect to the attached states $|\psi\rangle_3$ and $|\varphi\rangle_5$ over z and w . Observe that the relative factor of $(-1)^{\mathbf{p}_3 \cdot \mathbf{p}_5}$ between these two terms, which would normally be present, has already been taken care of by our definition (3.5) of the bra-vacuum, which includes the cocycle factors. The symmetry $z \leftrightarrow w$, $3 \leftrightarrow 5$, of the vertex, implies that we may apply Cauchy's theorem to find

$$\begin{aligned} & \oint_{|z|>|w|} \frac{dz}{2\pi i} \oint_0 \frac{dw}{2\pi i} \left\{ \mathcal{R}_1 V_{123}(z) |\psi\rangle_3 \overset{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) |\varphi\rangle_5 \right\} - \left(\begin{array}{c} z \leftrightarrow w \\ 3 \leftrightarrow 5 \end{array} \right) \\ &= \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} \left\{ \mathcal{R}_1 V_{123}(z) |\psi\rangle_3 \overset{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) |\varphi\rangle_5 \right\} \\ &= \oint_0 \frac{dw}{2\pi i} \left\{ \mathcal{R}_1 V_{146}(w) \overset{66'}{\cup} \oint_w \frac{dz}{2\pi i} \mathcal{R}_{6'} \bar{V}_{6'53}(z, w, w) |\psi\rangle_3 |\varphi\rangle_5 \right\}, \end{aligned} \quad (5.7)$$

where we have used duality of Eq. (4.30) (with the choice $y = w$) in the last step to reexpress the four-vertex; this derivation is the multistring analog of the calculation leading from (2.44) to (2.46). We repeat, however, that (5.7) is an operator on string Fock space (with two unsaturated legs, i.e. a “matrix”) rather than a state. As such we can now identify the four-vertex on the right with the contour integrals as the commutator of the integrated physical vertex operators associated with ψ and φ , respectively. Examining the transition functions $\bar{\tau}_{ij}$ for the vertex $\bar{V}^{[3]}(z, w, w)$ of Eqs. (4.34) and (4.35), we see that when $z = w$ they become independent of w . It follows that the residue of $\bar{V}^{[3]}(z, w, w)$ as $z \rightarrow w$ is independent of w . Consequently, the expression $\oint_w \frac{dz}{2\pi i} \bar{V}_{6'53}(z, w, w) |\psi\rangle_3 |\varphi\rangle_5$ is independent of w and hence the w integral sees only the first factor. This factor $\oint_0 \frac{dw}{2\pi i} \mathcal{R}_1 V_{146}(w)$ is the integrated vertex operator corresponding to the state on leg 6. We have in the above calculation considered the commutator of linear operators, but we have learnt that given the physical states ψ on leg 3 and φ on leg 5, we can assign to them the new state

$$[\psi, \varphi] = \oint_w \frac{dz}{2\pi i} \mathcal{R}_{6'} \bar{V}_{6'53}(z, w, w) |\psi\rangle_3 |\varphi\rangle_5, \quad (5.8)$$

which is nothing but Eq. (2.38) of Sect. 2.1 by virtue of the relation between $\bar{V}^{[3]}(z, w, w)$ and $V^{[3]}(z)$ of Eq. (4.43) and the fact that $V^{[3]}(z)$ has conformal weight one. In this way of viewing things, we calculate directly the commutator of two integrated vertex operators by sewing. The resulting vertex operator is associated with the state of Eq. (5.8). So we may equivalently think of the commutator in terms of a commutator of states and recover the viewpoint of Sect. 2. We conclude that (5.7) can be reexpressed as

$$\oint_{|z|>|w|} \frac{dz}{2\pi i} \oint_0 \frac{dw}{2\pi i} \left\{ \mathcal{R}_1 V_{123}(z) |\psi\rangle_3 \overset{22'}{\cup} \mathcal{R}_{2'} V_{2'45}(w) |\varphi\rangle_5 \right\} - \left(\begin{smallmatrix} z \leftrightarrow w \\ 3 \leftrightarrow 5 \end{smallmatrix} \right) = \oint_0 \frac{dw}{2\pi i} \mathcal{R}_1 V_{146}(w) |[\psi, \varphi]\rangle_6, \quad (5.9)$$

which corresponds to the last line of (2.46). We repeat that these arguments are valid only on-shell. Since the reader may be puzzled by the fact that we employed the three-vertex in (5.1) to define the commutator of states whereas in the discussion above we used four-vertices to introduce the commutator, we will now clarify this point. Suppose we put a (physical) state ξ on leg 2 in (5.9). Then we find the state $[[\psi, \varphi], \xi]$ on leg 1. This seems to contradict our definition (5.5) which would give the state $[\psi, [\varphi, \xi]]$ on leg 1 as the result of the scattering process. This superficial inconsistency is resolved by noticing that the left-hand side of (5.9) contains two terms both of which are of the form (5.5). Hence the identity (5.9) would lead to $[\psi, [\varphi, \xi]] - [\varphi, [\psi, \xi]] = [[\psi, \varphi], \xi]$ which is just the Jacobi identity! This means that applying the commutator of two integrated vertex operators to a state is the same as acting on the state with the integrated vertex operator associated with the commutator of states. Hence we are again (cf. the discussion after (2.39)) led to conclude that the integrated vertex operators realize the adjoint representation of the Lie algebra of states.

Just as before we can now iterate the above calculation to obtain a formula for the multiple commutator. We wish to evaluate

$$\oint_0 \frac{dz_3}{2\pi i} \oint_{|z_4|>|z_3|} \frac{dz_4}{2\pi i} \dots \oint_{|z_{N+1}|>|z_N|} \frac{dz_{N+1}}{2\pi i} \mathcal{R}_1 V^{[N+1]}(z_3, z_4, \dots, z_{N+1}) |\psi_3\rangle_3 |\psi_4\rangle_4 \dots |\psi_{N+1}\rangle_{N+1} \pm \dots, \quad (5.10)$$

where the dots stand for the remaining terms appearing in the multiple commutator with appropriate signs and reorderings of the contours. In analogy with the arguments leading from (2.44) to (2.45) we employ the generalization of locality (4.47) to arrive at the nested contour integral

$$\oint_0 \frac{dz_3}{2\pi i} \oint_{z_3} \frac{dz_4}{2\pi i} \dots \oint_{z_N} \frac{dz_{N+1}}{2\pi i} \mathcal{R}_1 V^{[N+1]}(z_3, z_4, \dots, z_{N+1}) |\psi_3\rangle_3 |\psi_4\rangle_4 \dots |\psi_{N+1}\rangle_{N+1}, \quad (5.11)$$

which again has two unsaturated legs. As before, we interpret the $(N+1)$ -vertex sandwiched with $|\psi_3\rangle_3 |\psi_4\rangle_4 \dots |\psi_{N+1}\rangle_{N+1}$ as the multiple commutator of the associated integrated physical vertex operators. Employing duality as discussed at the end of Sect. 4.2 we thus obtain

$$\begin{aligned} (5.10) &= \oint_0 \frac{dz_3}{2\pi i} \mathcal{R}_1 V_{12P'}(z_3) \overset{P'P}{\cup} \oint_{z_3} \frac{dz_4}{2\pi i} \dots \oint_{z_N} \frac{dz_{N+1}}{2\pi i} \mathcal{R}_P \bar{V}_{P34\dots N+1}^{[N]}(z_3, z_4, \dots, z_{N+1}) |\psi_3\rangle_3 |\psi_4\rangle_4 \dots |\psi_{N+1}\rangle_{N+1} \\ &= \oint_0 \frac{dz_3}{2\pi i} \mathcal{R}_1 V_{123}(z_3) |[\psi_{N+1}, [\psi_N, \dots, [\psi_4, \psi_3]] \dots]\rangle_3, \end{aligned} \quad (5.12)$$

where we have defined

$$\begin{aligned} &[\psi_{N+1}, [\psi_N, \dots, [\psi_4, \psi_3]] \dots] \\ &:= \oint_{z_3} \frac{dz_4}{2\pi i} \dots \oint_{z_N} \frac{dz_{N+1}}{2\pi i} \mathcal{R}_P \bar{V}_{P34\dots N+1}^{[N]}(z_3, z_4, \dots, z_{N+1}) |\psi_3\rangle_3 |\psi_4\rangle_4 \dots |\psi_{N+1}\rangle_{N+1}. \end{aligned} \quad (5.13)$$

This shows that the multiple commutator of integrated vertex operators yields a vertex operator which is associated with the above state on leg 3. Hence we may think of the multiple commutator of the states $\psi_3, \dots, \psi_{N+1}$ as given by this state and so recover the viewpoint of Sect. 2. Note that the two definitions (5.5) and (5.13) are completely equivalent. To see this we recall that V and \bar{V} agree on-shell so that we may insert for $\bar{V}_{P34\dots N+1}^{[N]}(z_3, z_4, \dots, z_{N+1})$ the expression $V_{P34\dots N+1}^{[N]}(z_4 - z_3, \dots, z_{N+1} - z_N)$ (cf. the diagram before (4.73)). Upon performing the shift of variables $z_i \rightarrow z'_i = z_i - z_{i-1}$ (for $4 \leq i \leq N+1$) the integrals become residues around zero and thus we indeed recover (5.5).

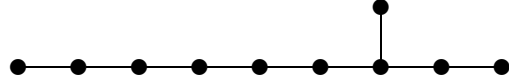
5.2 E_{10} : a brief review

Most of the information about the hyperbolic Kac Moody algebra E_{10} available in the mathematical literature can be gleaned from [38]; for more general information about hyperbolic (and indefinite) Kac Moody algebras, readers are advised to consult [37] as well as [46, 15, 18, 38, 55, 16, 45]. We will here briefly summarize [26, 25], whose notation and conventions we will adhere to, and where readers can find more details about the specific results needed here.

The simple roots of E_{10} are given by the lattice vectors

$$\begin{aligned}
\mathbf{r}_{-1} &= (0, 0, 0, 0, 0, 0, 0, 1, -1 | 0), \\
\mathbf{r}_0 &= (0, 0, 0, 0, 0, 0, 0, 1, -1, 0 | 0), \\
\mathbf{r}_1 &= (0, 0, 0, 0, 0, 0, 1, -1, 0, 0 | 0), \\
\mathbf{r}_2 &= (0, 0, 0, 0, 0, 1, -1, 0, 0, 0 | 0), \\
\mathbf{r}_3 &= (0, 0, 0, 0, 1, -1, 0, 0, 0, 0 | 0), \\
\mathbf{r}_4 &= (0, 0, 0, 1, -1, 0, 0, 0, 0, 0 | 0), \\
\mathbf{r}_5 &= (0, 1, -1, 0, 0, 0, 0, 0, 0, 0 | 0), \\
\mathbf{r}_6 &= (-1, -1, 0, 0, 0, 0, 0, 0, 0, 0 | 0), \\
\mathbf{r}_7 &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2}), \\
\mathbf{r}_8 &= (1, -1, 0, 0, 0, 0, 0, 0, 0, 0 | 0),
\end{aligned}$$

and correspond to the Dynkin diagram


(5.14)

The Cartan matrix is $A_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j$ ($-1 \leq i, j \leq 8$), where the scalar product is to be computed with the Minkowski metric ($+\dots + | -$) on the lattice. The root lattice of E_{10} generated by these simple roots is the unique even self-dual Lorentzian lattice $\Pi_{9,1}$ in ten dimensions (see [10] for details). In the following, the E_9 null root $\boldsymbol{\delta}$ will play an important role; it is

$$\boldsymbol{\delta} = \sum_{i=0}^8 n_i \mathbf{r}_i = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1 | 1), \quad (5.15)$$

where the coefficients n_i (called **marks** of E_9) can be read off from

$$\begin{bmatrix} & & & & & & 3 & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{bmatrix}. \quad (5.16)$$

The fundamental Weyl chamber \mathcal{C} of E_{10} is the convex cone generated by the **fundamental weights** $\boldsymbol{\Lambda}_i$ which obey $\boldsymbol{\Lambda}_i \cdot \mathbf{r}_j = -\delta_{ij}$ and are explicitly given by

$$\boldsymbol{\Lambda}_i = - \sum_{j=-1}^8 (A^{-1})_{ij} \mathbf{r}_j \quad \text{for } i = -1, 0, 1, \dots, 8 \quad (5.17)$$

in terms of the inverse Cartan matrix. Thus,

$$\boldsymbol{\Lambda} \in \mathcal{C} \quad \Longleftrightarrow \quad \boldsymbol{\Lambda} = \sum_{i=-1}^8 k_i \boldsymbol{\Lambda}_i \quad (5.18)$$

for $k_i \in \mathbf{Z}_+$. A special feature of E_{10} is that its root and weight lattices coincide as the lattice is self-dual. Since Weyl transformations preserve multiplicities and since every root can be brought into \mathcal{C} by means of a Weyl transformation, it is sufficient to consider only roots belonging to \mathcal{C} . Note also that the null root is just the first fundamental weight $\boldsymbol{\Lambda}_{-1} = \boldsymbol{\delta}$ and that all other fundamental roots are timelike. Hence the fundamental Weyl chamber lies entirely within the lightcone, touching it at precisely one edge. This is a very special feature of E_{10} , and also important for the DDF construction of [26], as otherwise one would have to deal with two or even more null directions at the same time.

A useful notion in the theory of hyperbolic Kac Moody algebras is the **level** ℓ of a given root $\boldsymbol{\Lambda}$, which is defined as the number of times the over-extended root \mathbf{r}_{-1} occurs in it (counted negatively for negative roots);

equivalently, it is the number of e_{-1} generators (for positive roots) or minus the number of f_{-1} generators (for negative roots) in the multiple commutator corresponding to the root Λ . A convenient formula for ℓ is

$$\ell \equiv \ell(\Lambda) := -\delta \cdot \Lambda \quad (5.19)$$

where δ is given in (5.15). Obviously, ℓ is not invariant under the full Weyl group $\mathcal{W}(E_{10})$ but only under its affine Weyl subgroup $\mathcal{W}(E_9)$. The level derives its importance from the fact that it grades the hyperbolic algebra with respect to its affine subalgebra. This is the content of the following important theorem [15, 16].

Theorem 5

Suppose that x is an element of E_{10} at level ℓ . Then it can be represented as a linear combination of $(\ell - 1)$ -fold commutators of level-one elements, viz.

$$x = [x_1, [x_2, \dots [x_{\ell-1}, x_\ell] \dots]] \quad (5.20)$$

where each x_i contains exactly one generator e_{-1} in the right-most position, i.e.

$$x_1 = [e_{i_1}, [e_{i_2}, \dots [e_{i_k}, e_{-1}] \dots]] \quad (5.21)$$

with $i_\nu \in \{0, 1, \dots, 8\}$, and similarly for the other x_i .

Specializing to E_{10} , we can thus write

$$E_{10} = \bigoplus_{\ell=-\infty}^{\infty} E_{10}^{[\ell]} \quad (5.22)$$

with

$$E_{10}^{[\ell]} = \underbrace{[E_{10}^{[1]}, [E_{10}^{[1]}, \dots, [E_{10}^{[1]}, E_{10}^{[1]}] \dots]]}_{(\ell-1) \text{ times}} \quad (5.23)$$

for positive ℓ where $E_{10}^{[\ell]}$ is the level- ℓ sector of the full algebra (for negative ℓ , one must take the contragredient representation generated by multiply commuting $E_{10}^{[-1]}$; this amounts to a simple replacement of all e_i 's by f_i 's). The level-zero sector $E_{10}^{[0]} = E_9$ is just the affine subalgebra, and the level-one elements $E_{10}^{[1]}$ are known to constitute the so-called basic representation [37, 38]. In terms of the graded decomposition (5.22) one can now in principle study the hyperbolic algebra level by level. All higher level sectors form (in general reducible) representations of the affine subalgebra, and since they can be generated through multiple commutators of level-one elements, it should be possible to understand them in terms of the irreducible representations obtained by taking multiple products of level-one representations. This idea has been successfully applied so far only to the level-two states, where it was possible to derive a general multiplicity formula [38, 15], but (to the best of our knowledge) not to level three and beyond. The problem here is that not all the representations appearing in the product occur in the Kac Moody algebra, because some of them drop out on account of the Serre relation $[e_{-1}, [e_{-1}, e_0]] = 0$. To identify the ones which are not present would require complete knowledge of the theory of irreducible representations of Kac Moody algebras of arbitrary level.

In [26] an attempt was made to tackle the problem from a different and at the same time more “physical” point of view by exploiting the vertex operator construction of affine Kac Moody algebras [32, 20], which associates the elements of the affine algebra with the physical string vertex operators for the emission of tachyons and photons. The modern formalism of vertex algebras [21, 5] relates the elements of a given indefinite (or generalized) Kac Moody algebra to the higher (massive) excitations of a string. A major advantage of this construction is that it not only affords a concrete realization of the abstract algebra but also that the L_0 physical state condition (2.18) immediately implies the Serre relations, so the corresponding elements of the (free) Lie algebra are eliminated from the outset. However, this does not mean that there is an “easy” realization of the hyperbolic algebra as we must now identify the states which are elements of the Lie algebra of physical states $\mathfrak{g}_{H_{9,1}}$ associated with the E_{10} root lattice, but *not* in E_{10} . Although the problem of isolating decoupled states bears some resemblance to the problem of discarding multiple commutators involving the Serre relations, one should realize that the decoupled states have nothing to do with the Lie algebra elements that must be discarded because of the Serre relations, as the latter never appear in the vertex algebra construction.

As explained in [26] all states in \mathfrak{g}_Λ can be represented in terms of DDF operators. The new, although not completely unexpected, feature is the relevance of longitudinal DDF for subcritical (i.e. rank $d < 26$) Kac Moody algebras. The longitudinal states decouple only for $d = 26$, in which case one is led to the “fake monster” $\mathfrak{g}_{E_{25,1}}$ of [6] (taking into account extra lightlike simple roots). In the DDF framework, the level-zero and level-one states are still relatively easy to understand: the affine subalgebra corresponds to the tachyon and photon states, and the level-one states can be reinterpreted as purely transversal states built on the tachyon state $|\mathbf{r}_{-1}\rangle$ (remember that \mathbf{r}_{-1} is the over-extended root) and its orbit under the action of the Weyl group $\mathcal{W}(E_9)$. Beyond level one it was shown in [26] by explicit computation that longitudinal states appear while at the same time certain transversal states decouple. The root spaces can then be understood rather explicitly in terms of polarization states for the corresponding root by analyzing which physical states cannot be reached via multiple commutators.

Let us now recall some basic features of the DDF construction. As is well known (see e.g. [56]), the DDF operators require for their definition a tachyon momentum \mathbf{v} (thus $\mathbf{v}^2 = 2$) and a lightlike vector $\mathbf{k} = \mathbf{k}(\mathbf{v})$ (thus $\mathbf{k}^2 = 0$) obeying $\mathbf{k} \cdot \mathbf{v} = 1$. For continuous momenta, such vectors can always be found and rotated into a convenient frame, but on the lattice this is in general not possible. For this reason we must extend the root lattice by rational points in order to apply the DDF construction in the present setting. To see this let us define the **DDF decomposition** of a given level ℓ root Λ by [26]

$$\Lambda = \mathbf{v} - n\mathbf{k} \quad (5.24)$$

where $\mathbf{k} \equiv \mathbf{k}(\mathbf{v}) := -\frac{1}{\ell}\delta$, and $n := 1 - \frac{1}{2}\Lambda^2$ is the number of steps required to reach the root by starting from \mathbf{v} and decreasing the momentum by \mathbf{k} at each step. Consequently, we have a factor ℓ^{-1} in the definition of \mathbf{k} for level ℓ , so clearly neither \mathbf{v} nor \mathbf{k} will in general be elements of lattice any more. To analyze the full algebra at *arbitrary* level we have to make use of the rational extension of the lattice (which is again self-dual, unlike the “intermediate” lattices for finite ℓ). In addition, we need a set of transversal polarization vectors $\xi^c = \xi^c(\mathbf{v}, \mathbf{k})$ subject to $\xi^c \cdot \mathbf{v} = \xi^c \cdot \mathbf{k} = 0$, which for convenience we assume to be orthonormalized. We here use letters $c, \dots = 1, \dots, 8$ from the beginning of the alphabet for the transversal indices as we reserve the letters i, j, \dots for the labeling of the one-string Fock spaces. Given these data, we can define the **transversal DDF operators** by (cf. [12])

$$A_m^c(\mathbf{v}) := \text{Res}_z \left[\xi^c \cdot \mathbf{P}(z) e^{im\mathbf{k}(\mathbf{v}) \cdot \mathbf{X}(z)} \right]. \quad (5.25)$$

These operators describe the emission of a photon with momentum $m\mathbf{k}$ and polarization ξ^c for the string. Note that no normal-ordering is required here. It is, however, required in the definition of **longitudinal DDF operators**, which we will also need and which are given by (cf. [8])

$$A_m^-(\mathbf{v}) := \mathcal{L}_m(\mathbf{v}) - \frac{1}{2} \sum_{a=1}^8 \sum_{n \in \mathbf{Z}} \times A_n^a(\mathbf{v}) A_{m-n}^a(\mathbf{v}) \times \quad (5.26)$$

with

$$\mathcal{L}_m(\mathbf{v}) := \text{Res}_z \left[\left(-\mathbf{v} \cdot \mathbf{P}(z) + \frac{1}{2} m \frac{d}{dz} \log(\mathbf{k} \cdot \mathbf{P}(z)) \right) e^{im\mathbf{k}(\mathbf{v}) \cdot \mathbf{X}(z)} \right] \quad (5.27)$$

and

$$\times A_m^a(\mathbf{v}) A_n^b(\mathbf{v}) \times := \begin{cases} A_m^a(\mathbf{v}) A_n^b(\mathbf{v}) & \text{if } m \leq n, \\ A_n^b(\mathbf{v}) A_m^a(\mathbf{v}) & \text{if } m > n, \end{cases} \quad (5.28)$$

where we used $\mathbf{v} \cdot \mathbf{k} = 1$. Unlike in [26] we will work with the operators A_m^- instead of \mathcal{L}_m , because they commute with the transversal DDF operators. In the sequel we will be careful to indicate the dependence of the DDF operators on their associated DDF momenta as we did above. In string theory these labels are usually omitted as one considers only a single set of DDF operators for a fixed pair (\mathbf{v}, \mathbf{k}) of DDF momenta, but here we will be required to simultaneously use *different* sets of DDF operators. Strictly speaking, we should even include the lightlike moment and write $A_m^a(\mathbf{v}, \mathbf{k})$. Here we can suppress the second label because for E_{10} all $\mathbf{k}(\mathbf{v})$ in the fundamental Weyl chamber are proportional to the null root δ , with the proportionality factor being unambiguously determined by \mathbf{v} .

In the remainder of this subsection, we shall provide some more details about the specific example studied in [26]. The particular root space analyzed there is associated with the level-two root $\Lambda = \Lambda_7$ which is dual to

the simple root \mathbf{r}_7 , and this will also be our principal example here. Explicitly, $\mathbf{\Lambda}_7$ is given by

$$\mathbf{\Lambda}_7 = \begin{bmatrix} & & & & & & 7 & & & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 & \end{bmatrix} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 2), \quad (5.29)$$

so $\mathbf{\Lambda}_7^2 = -4$. Its decomposition into two level-one tachyonic roots is $\mathbf{\Lambda}_7 = \mathbf{r} + \mathbf{s} + 2\boldsymbol{\delta}$, where

$$\begin{aligned} \mathbf{r} := \mathbf{r}_{-1} &= \begin{bmatrix} & & & & & & 0 & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{bmatrix} = (0, 0, 0, 0, 0, 0, 0, 1, -1 | 0), \\ \mathbf{s} &:= \begin{bmatrix} & & & & & & 1 & & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & \end{bmatrix} = (0, 0, 0, 0, 0, 0, 0, -1, -1 | 0). \end{aligned}$$

Since $n = 1 - \frac{1}{2}\mathbf{\Lambda}_7^2 = 3$ we have $\mathbf{\Lambda}_7 = \mathbf{v} - 3\mathbf{k}(\mathbf{v})$ with $\mathbf{k}(\mathbf{v}) := -\frac{1}{2}\boldsymbol{\delta}$ and

$$\mathbf{v} := \mathbf{r} + \mathbf{s} - \mathbf{k} = (0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2} | \frac{1}{2}).$$

Observe the extra factor of $\frac{1}{2}$ in $\mathbf{k}(\mathbf{v})$ as appropriate for level two, whereas we have $\mathbf{k}(\mathbf{r}) = \mathbf{k}(\mathbf{s}) = -\boldsymbol{\delta}$. As anticipated neither \mathbf{k} nor \mathbf{v} are elements of $\mathcal{H}_{9,1}$. Nevertheless, since $\mathbf{v} \cdot \mathbf{k} = 1$, the action of the DDF operators on the tachyonic ground-state $|\mathbf{v}\rangle$ is well defined. The three sets of transversal polarization vectors associated with the tachyon states $|\mathbf{r}\rangle$, $|\mathbf{s}\rangle$ and $|\mathbf{v}\rangle$ will be denoted by $\boldsymbol{\xi}^c \equiv \boldsymbol{\xi}^c(\mathbf{v})$, $\boldsymbol{\zeta}^a \equiv \boldsymbol{\zeta}^a(\mathbf{r})$ and $\boldsymbol{\eta}^b \equiv \boldsymbol{\eta}^b(\mathbf{s})$, respectively. Explicitly,

$$\begin{aligned} \boldsymbol{\xi}^1 &= \boldsymbol{\zeta}^1 = \boldsymbol{\eta}^1 := (1, 0, 0, 0, 0, 0, 0, 0, 0 | 0), \\ &\vdots \\ \boldsymbol{\xi}^7 &= \boldsymbol{\zeta}^7 = \boldsymbol{\eta}^7 := (0, 0, 0, 0, 0, 0, 0, 1, 0 | 0); \\ \boldsymbol{\zeta}^8 &:= (0, 0, 0, 0, 0, 0, 0, 1, 1 | 1), \\ \boldsymbol{\eta}^8 &:= (0, 0, 0, 0, 0, 0, 0, -1, 1 | 1), \\ \boldsymbol{\xi}^8 &:= (0, 0, 0, 0, 0, 0, 0, 1, 0 | 0). \end{aligned}$$

(the notation here is slightly different from [26]). In accordance with definitions (5.25) and (5.27), the associated DDF operators are designated by $A_m^c(\mathbf{v})$, $A_m^a(\mathbf{r})$ and $A_m^b(\mathbf{s})$, and $A_m^-(\mathbf{v})$, $A_m^-(\mathbf{r})$ and $A_m^-(\mathbf{s})$, respectively.

Beyond $\ell = 2$ longitudinal states appear, while certain transversal states decouple. All elements of $E_{10}^{(\mathbf{\Lambda}_7)}$ can be reached via the following commutators of level-one states and their $\mathcal{W}(E_9)$ rotated versions [26]:

$$\begin{aligned} &[A_{-1}^a(\mathbf{r})|\mathbf{r}\rangle, A_{-1}^b(\mathbf{s})|\mathbf{s}\rangle], \\ &[A_{-1}^a(\mathbf{r})A_{-1}^b(\mathbf{r})|\mathbf{r}\rangle, |\mathbf{s}\rangle], \\ &[A_{-2}^a(\mathbf{r})|\mathbf{r}\rangle, |\mathbf{s}\rangle]. \end{aligned} \quad (5.30)$$

The resulting states can be expressed in terms of DDF operators appropriate to the new groundstate $|\mathbf{v}\rangle$ as follows (no summation on repeated indices):

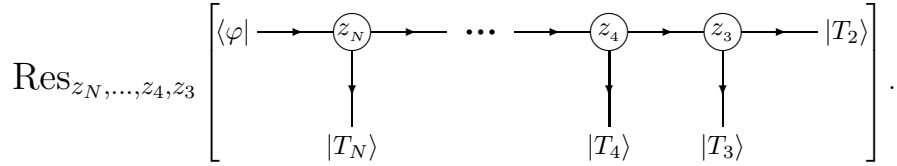
$$\begin{aligned} &A_{-2}^a(\mathbf{v})A_{-1}^b(\mathbf{v})|\mathbf{v}\rangle && \text{for } a, b \text{ arbitrary;} \\ &A_{-1}^a(\mathbf{v})A_{-1}^b(\mathbf{v})A_{-1}^c(\mathbf{v})|\mathbf{v}\rangle && \text{for } a \neq b, c; b \neq c; \\ &\left(A_{-3}^a(\mathbf{v}) - A_{-1}^a(\mathbf{v})A_{-1}^b(\mathbf{v})A_{-1}^b(\mathbf{v})\right)|\mathbf{v}\rangle && \text{for } a \neq b; \\ &\left(5A_{-3}^a(\mathbf{v})^i + A_{-1}^a(\mathbf{v})A_{-1}^a(\mathbf{v})A_{-1}^a(\mathbf{v})\right)|\mathbf{v}\rangle && \text{for } a \text{ arbitrary;} \\ &A_{-1}^a(\mathbf{v})A_{-2}^-(\mathbf{v})|\mathbf{v}\rangle && \text{for } a \text{ arbitrary;} \end{aligned} \quad (5.31)$$

Altogether, we get $64 + 2 \cdot 56 + 2 \cdot 8 = 192$ states in agreement with the formula in [38]. Despite the fact that this number coincides with the number of transversal states, we explicitly see the appearance of longitudinal as well as the disappearance of some transversal states. The orthogonal complement of $E_{10}^{(\mathbf{\Lambda}_7)}$ in $\mathbf{g}_{\mathcal{H}_{9,1}}^{(\mathbf{\Lambda}_7)}$ w.r.t. the usual string scalar product (2.98) (for which $(A_{-m}^a)^\dagger = A_m^a$) is spanned by the nine missing states

$$\begin{aligned} &A_{-3}^-(\mathbf{v})|\mathbf{v}\rangle, \\ &\left(2A_{-3}^a(\mathbf{v}) - 8A_{-1}^a(\mathbf{v})A_{-1}^a(\mathbf{v})A_{-1}^a(\mathbf{v}) + 3A_{-1}^a(\mathbf{v}) \sum_{b=1}^8 A_{-1}^b(\mathbf{v})A_{-1}^b(\mathbf{v})\right)|\mathbf{v}\rangle. \end{aligned} \quad (5.32)$$

These states *cannot* be reached by multiple commutation of the E_{10} Chevalley generators. Observe that neither (5.31) nor (5.32) are $SO(8)$ covariant.

Our main point in the remaining sections will be to demonstrate that these decoupled states can be directly and rather efficiently determined by means of the multistring vertices introduced in the foregoing sections. Thus, we here take the opposite approach to the one followed in [26]: instead of computing the root space elements (5.31) directly, we seek to identify the states (5.32) which are *not* in the root space by showing that they decouple from the string vertex. As we will see the overlap identities discussed before furnish the requisite tools for this task. In accordance with the representation (5.23), we represent the general level- ℓ element as an $(\ell - 1)$ -fold commutator of level-one elements. By formula (5.6) above, the corresponding string state can be alternatively viewed as the outcome of a string scattering process with ℓ incoming level-one states on the external legs of the string vertex, i.e. as the “out-state” emerging on the unsaturated leg (so this process involves a vertex with $N = \ell + 1$ external legs). The scattering process is depicted in the diagram below.



The crux of the matter is now that not every physical state at level ℓ can be produced by scattering level-one states (remember that these are just the transversal states associated with the tachyonic groundstate $|\mathbf{r}_{-1}\rangle$). Whether a given level- ℓ state φ decouples or not can be tested by attaching it to the above vertex: if φ decouples, this contraction must vanish for all possible choices of level-one states on the other legs. That is, φ must then obey the condition

$$V^{[\ell+1]}|\varphi\rangle_1|T_2\rangle_2|T_3\rangle_3\cdots|T_{\ell+1}\rangle_{\ell+1}=0, \quad (5.33)$$

where we have schematically denoted the transversal level-one states by $|T_i\rangle_i$, so $T_i \in E_{10}^{[1]}$ for all i . Although the relation (5.33) must be verified *for all* such states T_i , the task is substantially facilitated by momentum conservation which reduces the calculation to a finite number of checks for any given state φ at level ℓ . Note also that the total “level number” is conserved by the vertex; this is a consequence of momentum conservation along the axis defined by the null root δ .

In the remaining section we will exhibit the basic decoupling mechanism in specific examples. For this purpose we distinguish between longitudinal decoupling (for states φ involving longitudinal DDF operators) and transversal decoupling (for states φ involving only transversal DDF operators). As it turns out, the decoupling of longitudinal states is still relatively straightforward to analyze, and in particular does *not* depend on special properties of the lattice, though on its dimension (or rank) d ; in fact, decoupling remains valid in the continuum (with possible consequences for Liouville theory for $1 < d < 25$.) Transversal decoupling, on the other hand, is far more subtle, depending on the properties of the lattice in a crucial way, and having no continuum analog.

5.3 Decoupling of longitudinal states

Recall that in the course of the DDF construction we switched from the usual oscillator basis $\{\alpha_m^\mu\}$ to the DDF basis $\{A_m^a(\mathbf{v}), A_m^-(\mathbf{v}), L_m\}$ which nicely separates the Fock space into physical and unphysical states. For the discussion of decoupling of longitudinal states it turns out, however, that it is more useful to go over to the “non-diagonal” basis $\{A_m^a(\mathbf{v}), K_m(\mathbf{v}), L_m\}$ used in the proof of the no-ghost theorem (see [34]). Given DDF momenta \mathbf{v} and $\mathbf{k}(\mathbf{v})$ we take the usual set of transversal DDF oscillators, $A_m^a(\mathbf{v})$, and define

$$K_m(\mathbf{v}) := \mathbf{k} \cdot \boldsymbol{\alpha}_m \equiv k_\mu \alpha_m^\mu, \quad (5.34)$$

so that we have the commutation relations

$$[K_m(\mathbf{v}), K_n(\mathbf{v})] = 0, \quad [A_m^a(\mathbf{v}), K_n(\mathbf{v})] = 0, \quad [L_m, K_n(\mathbf{v})] = -nK_{m+n}(\mathbf{v}). \quad (5.35)$$

Thus both sets $\{A_m^-(\mathbf{v}), L_m\}$ and $\{K_m(\mathbf{v}), L_m\}$ commute with the transversal DDF operators, and the only difference is that the second set is not diagonal. Consequently, the K_n operators do not generate physical states.

The general form of a longitudinal DDF state in terms of the LK basis is

$$A_{-n_1}^-(\mathbf{v}) \dots A_{-n_N}^-(\mathbf{v})|\mathbf{v}\rangle = \left[\sum_{\substack{i_1 \geq \dots \geq i_I; j_1 \geq \dots \geq j_J \\ i_1 + \dots + j_J = n}} y_{i_1, \dots, i_I; j_1, \dots, j_J} L_{-i_1} \dots L_{-i_I} K_{-j_1}(\mathbf{v}) \dots K_{-j_J}(\mathbf{v}) \right. \\ \left. + \sum_{\substack{j_1 \geq \dots \geq j_J \\ j_1 + \dots + j_J = n}} x_{j_1, \dots, j_J} K_{-j_1}(\mathbf{v}) \dots K_{-j_J}(\mathbf{v}) \right] |\mathbf{v} - n\mathbf{k}\rangle, \quad (5.36)$$

where $n := n_1 + \dots + n_N$. The coefficients y and x are uniquely determined by the physical state conditions, i.e. by the requirement that the state is annihilated by the positive L_n 's. Note that each term involving only K_n 's (i.e. the second sum) is expected to vanish for $d = 26$ since we know from the no-ghost theorem that in the critical dimension the longitudinal states become null and hence no terms without Virasoro generators can occur.

Expressed in terms of the LK basis we find for the lowest longitudinal DDF states:

$$\begin{aligned} A_{-1}^-|\mathbf{v}\rangle &= -L_{-1}|\mathbf{v} - \mathbf{k}\rangle, \\ A_{-2}^-|\mathbf{v}\rangle &= \{-L_{-2} + 3L_{-1}K_{-1} + \frac{26-d}{4}[-K_{-2} + K_{-1}^2]\}|\mathbf{v} - 2\mathbf{k}\rangle, \\ A_{-3}^-|\mathbf{v}\rangle &= \{-L_{-3} + 4L_{-2}K_{-1} + \frac{5}{2}L_{-1}K_{-2} - \frac{17}{2}L_{-1}K_{-1}^2 \\ &\quad + \frac{2(26-d)}{3}[-K_{-3} + 3K_{-2}K_{-1} - 2K_{-1}^3]\}|\mathbf{v} - 3\mathbf{k}\rangle, \\ A_{-4}^-|\mathbf{v}\rangle &= \{-L_{-4} + 5L_{-3}K_{-1} + 3L_{-2}K_{-2} - 13L_{-2}K_{-1}^2 + \frac{7}{3}L_{-1}K_{-3} - 16L_{-1}K_{-2}K_{-1} + \frac{71}{3}L_{-1}K_{-1}^3 \\ &\quad + \frac{26-d}{8}[-10K_{-4} + 40K_{-3}K_{-1} + 13K_{-2}^2 - 86K_{-2}K_{-1}^2 + 43K_{-1}^4]\}|\mathbf{v} - 4\mathbf{k}\rangle, \\ A_{-2}^-A_{-2}^-|\mathbf{v}\rangle &= \{L_{-2}^2 - 3L_{-3}K_{-1} - 6L_{-2}L_{-1}K_{-1} + \frac{34-d}{2}L_{-2}K_{-2} + \frac{46-d}{2}L_{-2}K_{-1}^2 \\ &\quad + 9L_{-1}^2K_{-1}^2 + 5L_{-1}K_{-3} + \frac{3(44-d)}{2}L_{-1}K_{-2}K_{-1} + \frac{146-3d}{2}L_{-1}K_{-1}^3 \\ &\quad + \frac{26-d}{16}[-12K_{-4} + 48K_{-3}K_{-1} + (46-d)K_{-2}^2 - 2(82-d)K_{-2}K_{-1}^2 + (82-d)K_{-1}^4]\}|\mathbf{v} - 4\mathbf{k}\rangle, \end{aligned}$$

where we suppressed the \mathbf{v} dependence for notational convenience. We notice that A_{-1}^- leads to physical null states and should not be used for building physical states. Rewriting the A_m^- 's in terms of the LK basis quickly becomes rather cumbersome for higher excitations but we shall see below that we can circumvent most of this calculation because we need only the information about the x coefficients for decoupling.

We now consider the three-vertex with a longitudinal DDF state of the form (5.36) on leg 1 and arbitrary physical states on legs 2 and 3, respectively:

$$\theta \left(A_{-n_1}^{(1)-}(\mathbf{v}) \dots A_{-n_N}^{(1)-}(\mathbf{v})|\mathbf{v}\rangle_1 \right) \begin{array}{c} \xrightarrow{\quad} \textcircled{1} \xrightarrow{\quad} |\text{phys}\rangle_2 \\ \downarrow \\ |\text{phys}'\rangle_3 \end{array} \quad (5.37)$$

Note that in accordance with Eq. (5.3) we have not turned around leg 1 but have instead put in the Chevalley involution θ . Our aim is to derive by means of the overlap equations some simple criteria for the longitudinal DDF state to decouple. Since the A_m^- 's are integrated physical vertex operators we could invoke Eq. (3.21) to feed them straight through the vertex:

$$V^{[N]} \sum_{j=1}^N A_{-m}^{(j)-}(\mathbf{v}) = 0. \quad (5.38)$$

This identity, however, gives no insight into the decoupling mechanism. On the other hand, if we rewrite the longitudinal DDF in terms of the LK basis and move those through the vertex step by step then we do get powerful statements about the decoupling. First, we observe that all terms of the form $L_{-i_1} \dots L_{-i_k} K_{-i_{k+1}} \dots K_{-i_I} |\mathbf{v} - n\mathbf{k}\rangle$ vanish when fed through the vertex with physical states on the other legs. This follows from the discussion in Sect. 5.1 and the fact that the Chevalley involution commutes with the Virasoro generators. Thus only the

K terms contribute on-shell. As regards these terms we will next exploit the DDF construction and assume that the physical states on the other legs are transversal DDF states with lightlike vectors $\mathbf{k}', \mathbf{k}''$ which are proportional to the vector \mathbf{k} needed for $A_m^-(\mathbf{v}, \mathbf{k})$:

$$\theta \left(A_{-n_1}^{(1)-}(\mathbf{v}, \mathbf{k}) \dots A_{-n_N}^{(1)-}(\mathbf{v}, \mathbf{k}) | \mathbf{v} \rangle_1 \right) \longleftrightarrow \textcircled{1} \longrightarrow A_{-l_1}^{(2)a_1}(\mathbf{r}, \mathbf{k}'') \dots A_{-l_L}^{(2)a_L}(\mathbf{r}, \mathbf{k}'') | \mathbf{r} \rangle_2$$

$$\downarrow$$

$$A_{-m_1}^{(3)b_1}(\mathbf{s}, \mathbf{k}') \dots A_{-m_M}^{(3)b_M}(\mathbf{s}, \mathbf{k}') | \mathbf{s} \rangle_3$$
(5.39)

Before applying the overlap equation for the K_{-j} 's we note that

$$K_n(\mathbf{v}) A_{-l_1}^{a_1}(\mathbf{r}, \mathbf{k}'') \dots A_{-l_L}^{a_L}(\mathbf{r}, \mathbf{k}'') | \mathbf{r} \rangle = 0,$$

$$K_n(\mathbf{v}) A_{-m_1}^{b_1}(\mathbf{s}, \mathbf{k}') \dots A_{-m_M}^{b_M}(\mathbf{s}, \mathbf{k}') | \mathbf{s} \rangle = 0,$$
(5.40)

for $n > 0$ because of $\mathbf{k} \cdot \mathbf{k}'' = \mathbf{k} \cdot \boldsymbol{\zeta}^{a_i} = 0$ and $\mathbf{k} \cdot \mathbf{k}' = \mathbf{k} \cdot \boldsymbol{\eta}^{b_j} = 0$, respectively⁷. In view of the overlap Eq. (4.54) we therefore conclude that any K_{-j} , when fed through the vertex, becomes $K_0^{(3)}(\mathbf{v}) \equiv \mathbf{k} \cdot \boldsymbol{\alpha}_0^{(3)}$ acting on $| \mathbf{s} \rangle_3$ (note that we pick up one minus sign from the Chevalley involution and one from the overlaps giving the stated result). Thus it is just a *number* $\kappa := \mathbf{k} \cdot \mathbf{s}$ (or $\mathbf{k}' = \kappa \mathbf{k}$) and the net result of feeding a product of longitudinal DDF operators through the vertex is a *polynomial* $P(\kappa)$ which we will refer to as **(longitudinal) decoupling polynomial**. We will write $P_{n_1, \dots, n_N}^{[3]}(\kappa)$ for the polynomial encoding the decoupling of the longitudinal DDF state $A_{-n_1}^- \dots A_{-n_N}^- | \mathbf{v} \rangle$ from the three-vertex. Using the decomposition in Eq. (5.36) it has the general form

$$P_{n_1, \dots, n_N}^{[3]}(\kappa) := \sum_{\substack{j_1 \geq \dots \geq j_J \\ j_1 + \dots + j_J = n}} x_{j_1, \dots, j_J} \kappa^J.$$
(5.41)

This allows us to rewrite the result of the above diagram as

$$P_{n_1, \dots, n_N}^{[3]}(\kappa) \cdot \left[\begin{array}{c} | -\mathbf{v} + n\mathbf{k} \rangle_1 \longleftrightarrow \textcircled{1} \longrightarrow A_{-l_1}^{(2)a_1}(\mathbf{r}, \mathbf{k}'') \dots A_{-l_L}^{(2)a_L}(\mathbf{r}, \mathbf{k}'') | \mathbf{r} \rangle_2 \\ \downarrow \\ A_{-m_1}^{(3)b_1}(\mathbf{s}, \mathbf{k}') \dots A_{-m_M}^{(3)b_M}(\mathbf{s}, \mathbf{k}') | \mathbf{s} \rangle_3 \end{array} \right].$$
(5.42)

After some algebra, we arrive at the following list of decoupling polynomials for $\sum_i n_i \leq 7$:

$$P_2^{[3]}(\kappa) = \frac{26-d}{4} \kappa (\kappa - 1),$$
(5.43)

$$P_3^{[3]}(\kappa) = -\frac{2(26-d)}{3} \kappa (\kappa - 1) (2\kappa - 1),$$
(5.44)

$$P_4^{[3]}(\kappa) = \frac{26-d}{8} \kappa (\kappa - 1) (43\kappa^2 - 43\kappa + 10),$$
(5.45)

$$P_{2,2}^{[3]}(\kappa) = \frac{26-d}{16} \kappa (\kappa - 1) [(82-d)\kappa^2 - (82-d)\kappa + 12],$$
(5.46)

$$P_5^{[3]}(\kappa) = -\frac{26-d}{3} \kappa (\kappa - 1) (2\kappa - 1) (29\kappa^2 - 29\kappa + 6),$$
(5.47)

$$P_{3,2}^{[3]}(\kappa) = -\frac{26-d}{30} \kappa (\kappa - 1) (2\kappa - 1) [(558-5d)\kappa^2 - (558-5d)\kappa + 84],$$
(5.48)

$$P_6^{[3]}(\kappa) = \frac{26-d}{24} \kappa (\kappa - 1) [1568\kappa^4 - 3136\kappa^3 + 2237\kappa^2 - 669\kappa + 70],$$
(5.49)

$$P_{4,2}^{[3]}(\kappa) = \frac{26-d}{288} \kappa (\kappa - 1) [(54686 - 387d)\kappa^4 - (109372 - 774d)\kappa^3 + (75754 - 477d)\kappa^2 - (21068 - 90d)\kappa + 1920],$$
(5.50)

$$P_{3,3}^{[3]}(\kappa) = \frac{26-d}{72} \kappa (\kappa - 1) [(8773 - 128d)\kappa^4 - (17546 - 256d)\kappa^3$$

⁷This is a point where the construction could go wrong if the fundamental Weyl chamber contained more than one null direction, because the products of the null vectors and the polarization vectors would then no longer vanish in general.

$$+ (11819 - 160d)\kappa^2 - (3046 - 32d)\kappa + 216], \quad (5.51)$$

$$P_{2,2,2}^{[3]}(\kappa) = -\frac{26-d}{192}\kappa(\kappa-1)[(34460 - 660d + 3d^2)\kappa^4 - (68920 - 1320d + 6d^2)\kappa^3 \\ + (45076 - 768d + 3d^2)\kappa^2 - (10616 - 108d)\kappa + 768], \quad (5.52)$$

$$P_7^{[3]}(\kappa) = -\frac{26-d}{60}\kappa(\kappa-1)(2\kappa-1)[6367\kappa^4 - 12734\kappa^3 + 8873\kappa^2 - 2506\kappa + 240], \quad (5.53)$$

$$P_{5,2}^{[3]}(\kappa) = -\frac{26-d}{420}\kappa(\kappa-1)(2\kappa-1)[(173602 - 1015d)\kappa^4 - (347204 - 2030d)\kappa^3 \\ + (236278 - 1225d)\kappa^2 - (62676 - 210d)\kappa + 5400], \quad (5.54)$$

$$P_{4,3}^{[3]}(\kappa) = -\frac{26-d}{420}\kappa(\kappa-1)(2\kappa-1)[(125851 - 1505d)\kappa^4 - (251702 - 3010d)\kappa^3 \\ + (167909 - 1855d)\kappa^2 - (42058 - 350d)\kappa + 3000], \quad (5.55)$$

$$P_{3,2,2}^{[3]}(\kappa) = -\frac{26-d}{120}\kappa(\kappa-1)(2\kappa-1)[(94916 - 1396d + 5d^2)\kappa^4 - (189832 - 2792d + 10d^2)\kappa^3 \\ + (124364 - 1624d + 5d^2)\kappa^2 - (29448 - 228d)\kappa + 2160]. \quad (5.56)$$

We make the following important observations.

All decoupling polynomials vanish identically in the critical dimension $d = 26$ reflecting the decoupling of all longitudinal states there, in accord with the no-ghost theorem.

In general, decoupling takes only place at certain values of κ , namely at those zeros of the polynomials which are of the form $\kappa = \frac{1}{\ell}$ for $\ell \in \mathbf{N}$. In view of diagram (5.42) one might argue that this only constitutes a sufficient criterion for decoupling and in general the remaining diagram could also account for decoupling. But this is not the case since the oscillator part of the longitudinal state, which is the relevant piece of information for decoupling, has been completely absorbed into the decoupling polynomial and this is *independent* of the choice of transversal level-one states on the other legs. Of course, the remaining diagram could vanish for a special choice of transversal states, but for decoupling it had to vanish for arbitrary transversal level-one states on legs 2 and 3.

Once a “purely” longitudinal DDF state (i.e. involving only A_m^- ’s) decouples then also the whole transversal Heisenberg module built on it will also decouple because the transversal DDF oscillators commute with the LK basis. This shows that the question of missing longitudinal states can be completely solved in terms of the decoupling polynomials.

By construction, the decoupling polynomials have vanishing constant term, i.e. have a zero at $\kappa = 0$. But $\mathbf{k} \cdot \mathbf{s} = 0$ means that the state on leg 3 is an element of the E_9 subalgebra. Since, by level conservation, we then must have a level-one element on leg 2 we can interpret the zero at $\kappa = 0$ as the fact that the transversal level-one DDF states form a representation of E_9 . In view of the antisymmetry of the Lie bracket, interchanging legs 2 and 3 should make no difference for the decoupling, so that we also expect a zero at $\kappa = 1$. Indeed, the above examples show this feature and we will give a general proof below.

Inspection of the list of examples tempts us to conjecture a generic zero at $\kappa = \frac{1}{2}$ for odd total mode number, i.e.

$$P_{n_1, \dots, n_N}^{[3]}(\frac{1}{2}) = 0 \quad \text{for } n_1 + \dots + n_N \text{ odd.} \quad (5.57)$$

In other words, no (purely) longitudinal DDF states with total odd mode number can occur as level-two elements of E_{10} . This is in agreement with our example Λ_7 described in the last section where we found $A_{-3}^-(\mathbf{v})|\mathbf{v}\rangle$ not to be an element of E_{10} . Below, we will indicate how to prove the conjecture in general.

So far, we have always referred to E_{10} in our discussion of decoupling. But in the derivation of the decoupling polynomials we actually did not use the fact that the momenta lie on a lattice but merely exploited features of the old DDF construction which are of course also valid for the continuum case. Thus the pattern of longitudinal decoupling as presented above also applies to *any* bosonic string model. In particular, we can now analyze other hyperbolic Kac Moody algebras if we take the corresponding root lattice as momentum lattice for the string.

It is clear that the decoupling polynomial in (5.41) has degree n at most. Taking further into account the universal factor $\kappa(\kappa-1)$, there can be at most $n-1$ linearly independent decoupling polynomials. For fixed n , on the other hand, we know that there are $p_1(n) - p_1(n-1)$ physical longitudinal DDF states and hence the same number of decoupling polynomials. In other words, whereas the number of longitudinal DDF states grows exponentially the number of independent decoupling polynomials increases linearly. This means that there is a wealth of decoupling of longitudinal states. In fact, for large n (which corresponds to highly excited massive states) almost all longitudinal states decouple and only a small fraction leads to E_{10} elements. This feature

explains, for example, why the level-two multiplicity formula for E_{10} found in [38] stays close to the number of transversal states with slowly increasing deviation. But this argument will also apply in a more general context to all hyperbolic Kac Moody algebras. There, the common feature of all explicitly known root multiplicities is that they agree with or lie very near to the number of transversal states (see e.g. [37] for a table of root multiplicities of the hyperbolic extension of $A_1^{(1)}$). Longitudinal decoupling thus gives a qualitative explanation of this phenomenon.

It is possible to extract some information about the general structure of decoupling polynomials from the physical state conditions. More specifically, let us consider the implications of the condition that a longitudinal DDF state of the form (5.36) is annihilated by L_1 :

$$0 \stackrel{!}{=} \left\{ \sum_{\substack{i_1, \dots, i_I \\ i_1 + \dots + i_I = n}} y_{i_1, \dots, i_I} \left[(1 + i_1) L_{1-i_1} L_{-i_2} \dots L_{-i_k} K_{-i_{k+1}} \dots K_{-i_I} + \dots \right. \right. \\ + (1 + i_k) L_{-i_1} \dots L_{-i_{k-1}} L_{1-i_k} K_{-i_{k+1}} \dots K_{-i_I} \\ + i_{k+1} L_{-i_1} \dots L_{-i_k} K_{1-i_{k+1}} K_{-i_{k+2}} \dots K_{-i_I} + \dots \\ \left. \left. + i_I L_{-i_1} \dots L_{-i_k} K_{-i_{k+1}} \dots K_{-i_{I-1}} K_{1-i_I} \right] \right\} | \mathbf{v} - n \mathbf{k} \rangle.$$

In order to fulfill this condition the coefficients of all linearly independent combinations of LK basis elements have to vanish. Let us concentrate on terms which involve only K 's. We claim that they can only come from the second sum. Indeed, terms of the form $L_{-1}K_{-i_2} \dots K_{-i_l}|\mathbf{v} - n\mathbf{k}\rangle$ with $i_2 \geq \dots \geq i_l$ and $i_2 + \dots + i_l = n - 1$ do not give rise to pure K monomials since $[L_1, L_{-1}] = 2L_0$ and $L_0K_{-i_2} \dots K_{-i_l}|\mathbf{v} - n\mathbf{k}\rangle = 0$. Thus we arrive at the following necessary condition:

$$0 \stackrel{!}{=} \sum_{\substack{j_1 \geq \dots \geq j_J \\ j_1 + \dots + j_J = n}} x_{j_1, \dots, j_J} \left[j_1 K_{1-j_1} K_{-j_2} \dots K_{-j_J} + \dots + j_J K_{-j_1} \dots K_{-j_{J-1}} K_{1-j_J} \right], \quad (5.58)$$

where we put $K_0 \equiv 1$ due to $\mathbf{k} \cdot (\mathbf{v} - n\mathbf{k}) = 1$. This already severely constrains the L independent part of a longitudinal physical state for it yields $p_1(n-1)$ linear equations for the $p_1(n)$ coefficients x_{j_1, \dots, j_J} . Consequently it also determines a major part of the decoupling polynomials. For example, if we add up all linear equations we obtain

$$\begin{aligned}
0 &= \sum_{\substack{j_1 \geq \dots \geq j_J \\ j_1 + \dots + j_J = n}} x_{j_1, \dots, j_J} [j_1 + \dots + j_J] \\
&= n \sum_{\substack{j_1 \geq \dots \geq j_J \\ j_1 + \dots + j_J = n}} x_{j_1, \dots, j_J},
\end{aligned} \tag{5.59}$$

but, in view of Eq. (5.41), this is nothing but the statement that the decoupling polynomial vanishes at $\kappa = 1$, i.e. we can indeed always pull out the factor $(\kappa - 1)$. We have also checked for the above list of decoupling polynomials that, for odd n , the zeros at $\kappa = \frac{1}{2}$ also follow from the information encoded in Eq. (5.58), but we have not yet succeeded in finding a general proof. Note that the condition (5.58) is “universal” in the sense that (apart from the factor $(26 - d)$ inside) it is independent of d and thus the same for all subcritical strings.

If we want to probe higher-level decoupling we encounter a new subtlety: the decoupling polynomials will depend on the Koba Nielsen variables as parameters. The reason for this is that we can always only fix three Koba Nielsen points whereas there are N of them for the N -vertex. Using the overlap equations (4.74) we conclude that any $K_{-m}^{(1)}$, when fed through the vertex, becomes $\sum_{i=3}^N z_i^m K_0^{(i)}$ so that the decoupling polynomial for the process

$$\theta(A_{-n_1}^{(1)-}(\mathbf{v}, \mathbf{k}) \dots A_{-n_N}^{(1)-}(\mathbf{v}, \mathbf{k}) | \mathbf{v} \rangle_1) \longrightarrow \begin{array}{c} \text{---} \circ z_i \text{---} \\ \downarrow \quad \nearrow \quad \nearrow \\ |T_2\rangle_2 \\ |T_3\rangle_3 \\ \vdots \\ |T_N\rangle_N \end{array} \quad (5.60)$$

is given by

$$P_{n_1, \dots, n_N}^{[N]}(\kappa) := \sum_{\substack{m_1 \geq \dots \geq m_J \\ m_1 + \dots + m_J = n}} \sum_{i_1, \dots, i_J=3}^N x_{m_1, \dots, m_J}(z_{i_1})^{m_1} \dots (z_{i_J})^{m_J} \kappa^J, \quad (5.61)$$

with $n := n_1 + \dots + n_N$.

5.4 On the decoupling of transversal states

We now turn to the decoupling of transversal states, which is our most striking result. It is here that the special properties of the lattice enter in a crucial and still mysterious way, as there is no analogous decoupling in the continuum, unlike for the longitudinal states discussed in the preceding section. Due to our lack of general understanding we will in this section limit ourselves to the discussion of the specific example corresponding to the root space $E_{10}^{(A_7)}$. For the explicit calculations we will mostly work with the vertex V^{CSV} whose cyclic symmetry implies the cyclic symmetry of the associated overlap equations, which were given in Sect. 4.3. We will also need the expressions of the DDF states entering (5.30) in terms of oscillators. Beside the obvious formulas

$$A_{-1}^a(\mathbf{r})|\mathbf{r}\rangle = (\zeta^a \cdot \alpha_{-1})|\mathbf{r} + \delta\rangle, \quad A_{-1}^b(\mathbf{s})|\mathbf{s}\rangle = (\eta^b \cdot \alpha_{-1})|\mathbf{s} + \delta\rangle, \quad (5.62)$$

we shall need

$$A_{-2}^a(\mathbf{r})|\mathbf{r}\rangle = \left\{ (\zeta^a \cdot \alpha_{-2}) + 2(\zeta^a \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) \right\} |\mathbf{r} + 2\delta\rangle \quad (5.63)$$

as well as

$$\begin{aligned} A_{-3}^c(\mathbf{v})|\mathbf{v}\rangle = & \left\{ (\xi^c \cdot \alpha_{-3}) + \frac{3}{2}(\xi^c \cdot \alpha_{-2})(\delta \cdot \alpha_{-1}) + \frac{3}{4}(\xi^c \cdot \alpha_{-1})(\delta \cdot \alpha_{-2}) \right. \\ & \left. + \frac{9}{8}(\xi^c \cdot \alpha_{-1})(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) \right\} |\mathbf{v} + \frac{3}{2}\delta\rangle \end{aligned} \quad (5.64)$$

and

$$\begin{aligned} A_{-1}^c(\mathbf{v})A_{-1}^d(\mathbf{v})A_{-1}^e(\mathbf{v})|\mathbf{v}\rangle = & \left\{ (\xi^c \cdot \alpha_{-1})(\xi^d \cdot \alpha_{-1})(\xi^e \cdot \alpha_{-1}) \right. \\ & \left. + \frac{3}{2}\delta^{(cd)}(\xi^e \cdot \alpha_{-1})\left[\frac{1}{4}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) + \frac{1}{2}(\delta \cdot \alpha_{-2})\right] \right\} |\mathbf{v} + \frac{3}{2}\delta\rangle, \end{aligned} \quad (5.65)$$

where (cde) means symmetrization in the indices c, d, e with strength one. These formulas can be read off from the appendix of [26]; the only point to remember is that because $\mathbf{k}(\mathbf{r}) = \mathbf{k}(\mathbf{s}) = -\delta$ and $\mathbf{k}(\mathbf{v}) = -\frac{1}{2}\delta$, we must replace δ by $\frac{1}{2}\delta$ everywhere in the appropriate formulas of [26] to get (5.64) and (5.65). Furthermore, we will need to make use of the Chevalley involuted states obtained by replacing α_m^μ and $|\mathbf{v}\rangle$ by $-\alpha_m^\mu$ and $|\mathbf{-v}\rangle$, respectively, in the above expressions, cf. Eq. (2.96).

To prove that a given state ψ decouples, we will have to show that expressions like

$$V^{[3]} \left\{ |\psi\rangle_1 \otimes A_{-1}^{(2)a}(\mathbf{r})|\mathbf{r}\rangle_2 \otimes A_{-1}^{(3)b}(\mathbf{s})|\mathbf{s}\rangle_3 \right\}, \quad (5.66)$$

$$V^{[3]} \left\{ |\psi\rangle_1 \otimes A_{-2}^{(2)a}(\mathbf{r})|\mathbf{r}\rangle_2 \otimes |\mathbf{s}\rangle_3 \right\}, \quad (5.67)$$

$$V^{[3]} \left\{ |\psi\rangle_1 \otimes A_{-1}^{(2)a}(\mathbf{r})A_{-1}^{(2)b}(\mathbf{r})|\mathbf{r}\rangle_2 \otimes |\mathbf{s}\rangle_3 \right\}, \quad (5.68)$$

vanish for all possible choices of the transversal indices⁸. Since we will only attach physical states to the legs, any three-vertex could be used. However, for all practical calculations, and in particular those involving oscillator overlaps, we will make use of the CSV vertex V^{CSV} . Before proceeding with the calculation, however, we note

⁸We here write out the tensor product symbol \otimes unlike in previous formulas in order to render the expressions more transparent.

an important simplification due to the following identity which is a special case of (3.21) and the transversal analog of (5.38):

$$V^{[3]} \sum_{j=1}^3 A_m^{(j)a}(\mathbf{r}) = 0. \quad (5.69)$$

It is crucial here that the DDF operators refer to the same DDF momenta although they act on states in different Fock spaces which in general are constructed on other tachyonic groundstates with different DDF momenta, and we therefore must exercise some care to indicate precisely which DDF operators are meant. As an example let us apply the above identity to the two-string state $A_{-1}^{(2)a}(\mathbf{r})|\mathbf{r}\rangle_2 \otimes A_{-1}^{(3)b}(\mathbf{s})|\mathbf{s}\rangle_3$, so the first leg is not saturated. We get

$$V^{[3]} \left\{ A_{-1}^{(2)a}(\mathbf{r})|\mathbf{r}\rangle_2 \otimes A_{-1}^{(3)b}(\mathbf{s})|\mathbf{s}\rangle_3 \right\} = V^{[3]} \left\{ |\mathbf{r}\rangle_2 \otimes A_{-1}^{(3)a}(\mathbf{r})A_{-1}^{(3)b}(\mathbf{s})|\mathbf{s}\rangle_3 + |\mathbf{r}\rangle_2 \otimes A_{-1}^{(3)b}(\mathbf{s})|\mathbf{s}\rangle_3 A_{-1}^{(1)a}(\mathbf{r}) \right\}. \quad (5.70)$$

The first term on the right-hand side tells us that the expressions (5.66) and (5.68) are related. The last term on the right-hand side of (5.70) contains a DDF operator acting on the first leg; when this operator acts on the Chevalley involuted state $|- \mathbf{v}\rangle$, we obtain

$$A_{-m}^a(\mathbf{r})|- \mathbf{v}\rangle = \text{Res}_z \left[\zeta^a \cdot \mathbf{P}(z) e^{i2m\mathbf{k}(-\mathbf{v}) \cdot \mathbf{X}(z)} \right] |- \mathbf{v}\rangle \equiv \zeta_\mu^a A_{2m}^\mu(-\mathbf{v})|- \mathbf{v}\rangle, \quad (5.71)$$

because $\mathbf{k}(-\mathbf{v}) = +\frac{1}{2}\boldsymbol{\delta} = -\frac{1}{2}\mathbf{k}(\mathbf{r})$ (note the contraction with the polarization vector ζ_μ^a on the right-hand side whereas the operator $A_{2m}^a(-\mathbf{v})$ is defined with ξ_μ^a , which differs from ζ_μ^a for $a = 8$). Thus the creation operator is converted into an annihilation operator whose index is multiplied by two (for higher levels, the index will be similarly multiplied by ℓ). Below we will apply this identity to a state built with odd-moded DDF operators on the tachyonic groundstate, and so $A_{-m}^{(1)a}(\mathbf{r})$ will commute through to annihilate the state. Consequently, the second term on the right-hand side of (5.70) vanishes in this case, and the expressions (5.66) and (5.68) are, in fact, equivalent.

To evaluate (5.66) and (5.67) in practice, we first express the states on legs 2 and 3 in terms of oscillators according to (5.62) and (5.63), and then feed through the creation operators α_{-n} (with $n > 0$) to the other legs where they become annihilation or momentum operators acting on the state ψ by the oscillator overlap equations (4.54). Next we commute these α_n 's (with $n \geq 0$) through the creation operators defining the state ψ until they hit the oscillator vacuum. If any creation operators are still left on any of the legs, we repeat this process until all creation operators have been eliminated. The result of the calculation is some polynomial involving products of the polarization vectors and DDF momenta in various combinations multiplying

$$V^{[3]} |\mathbf{t}_1\rangle_1 \otimes |\mathbf{t}_2\rangle_2 \otimes |\mathbf{t}_3\rangle_3 = \delta_{\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3, 0}, \quad (5.72)$$

where $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ are the momenta of the three states attached to the three legs (in the example to be discussed, we have $\mathbf{t}_1 = -\mathbf{r} - \boldsymbol{\delta}$, $\mathbf{t}_2 = -\mathbf{s} - \boldsymbol{\delta}$ and $\mathbf{t}_3 = \mathbf{v} + \frac{3}{2}\boldsymbol{\delta}$). Clearly, this result just expresses momentum conservation. It also shows why we have to apply the Chevalley involution to some of the states, since the scalar product evidently vanishes if all three momenta correspond to positive roots. In other words we have to arrange for the momenta (i.e. the roots) to respect momentum conservation, which leaves only a finite number of states to be checked. To establish the decoupling it then remains to show that the prefactor of (5.72) vanishes for all possible choices of polarizations.

We will now show how this works for the above example with the vertex V^{CSV} , using the Chevalley involuted version of (5.62) on legs 2 and 3. Leaving the state ψ on the first leg arbitrary for the moment (it will be specified below), we get

$$\begin{aligned} & V^{\text{CSV}} \left\{ |\psi\rangle_1 \otimes (\zeta^a \cdot \alpha_{-1}^{(2)})|-\mathbf{r} - \boldsymbol{\delta}\rangle_2 \otimes (\eta^b \cdot \alpha_{-1}^{(3)})|-\mathbf{s} - \boldsymbol{\delta}\rangle_3 \right\} \\ &= V^{\text{CSV}} \left\{ (\zeta^a \cdot \eta^b)|\psi\rangle_1 \otimes |-\mathbf{r} - \boldsymbol{\delta}\rangle_2 \otimes |-\mathbf{s} - \boldsymbol{\delta}\rangle_3 \right. \\ &\quad \left. + (\zeta^a \cdot \alpha_1^{(1)} - \zeta^a \cdot \alpha_0^{(1)})|\psi\rangle_1 \otimes |-\mathbf{r} - \boldsymbol{\delta}\rangle_2 \otimes (\eta^b \cdot \alpha_{-1}^{(3)})|-\mathbf{s} - \boldsymbol{\delta}\rangle_3 \right\} \\ &= V^{\text{CSV}} \left\{ \left[(\zeta^a \cdot \eta^b) + \left(\eta^b \cdot \mathbf{r} + \sum_{n=1}^{\infty} \eta^b \cdot \alpha_n^{(1)} \right) (\zeta^a \cdot \alpha_1^{(1)} - \zeta^a \cdot \alpha_0^{(1)}) \right] |\psi\rangle_1 \otimes |-\mathbf{r} - \boldsymbol{\delta}\rangle_2 \otimes |-\mathbf{s} - \boldsymbol{\delta}\rangle_3 \right\}, \quad (5.73) \end{aligned}$$

where we have used the overlaps (4.54) for $\alpha_{-1}^{(2)}$ and $\alpha_{-1}^{(3)}$, respectively. We now take ψ to be either $A_{-3}^c(\mathbf{v})|\mathbf{v}\rangle$ or $A_{-1}^c(\mathbf{v})A_{-1}^d(\mathbf{v})A_{-1}^e(\mathbf{v})|\mathbf{v}\rangle$ whose form in terms of oscillators is given in (5.64) and (5.65), respectively. After some oscillator algebra we obtain

$$\begin{aligned} & \left[(\zeta^a \cdot \eta^b) + \left(\eta^b \cdot \mathbf{r} + \sum_{n=1}^{\infty} \eta^b \cdot \alpha_n \right) (\zeta^a \cdot \alpha_1 - \zeta^a \cdot \alpha_0) \right] A_{-3}^c(\mathbf{v})|\mathbf{v}\rangle \\ &= \left\{ [(\zeta^a \cdot \eta^b) - (\eta^b \cdot \mathbf{r})(\zeta^a \cdot \mathbf{v})] [(\xi^c \cdot \alpha_{-3}) + \frac{3}{2}(\xi^c \cdot \alpha_{-2})(\delta \cdot \alpha_{-1}) + \frac{3}{4}(\xi^c \cdot \alpha_{-1})(\delta \cdot \alpha_{-2}) \right. \\ & \quad \left. + \frac{9}{8}(\xi^c \cdot \alpha_{-1})(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1})] \right. \\ & \quad \left. + [(\zeta^a \cdot \xi^c)(\eta^b \cdot \mathbf{r}) - (\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^c)] \left[\frac{3}{4}(\delta \cdot \alpha_{-2}) + \frac{9}{8}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) \right] \right. \\ & \quad \left. - 3(\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^c) [(\delta \cdot \alpha_{-1}) + 1] \right\} |\mathbf{v} + \frac{3}{2}\delta\rangle \end{aligned} \quad (5.74)$$

for (5.64) and

$$\begin{aligned} & \left[(\zeta^a \cdot \eta^b) + \left(\eta^b \cdot \mathbf{r} + \sum_{n=1}^{\infty} \eta^b \cdot \alpha_n \right) (\zeta^a \cdot \alpha_1 - \zeta^a \cdot \alpha_0) \right] A_{-1}^c(\mathbf{v})A_{-1}^d(\mathbf{v})A_{-1}^e(\mathbf{v})|\mathbf{v}\rangle \\ &= \left\{ [(\zeta^a \cdot \eta^b) - (\eta^b \cdot \mathbf{r})(\zeta^a \cdot \mathbf{v})] [(\xi^c \cdot \alpha_{-1})(\xi^d \cdot \alpha_{-1})(\xi^e \cdot \alpha_{-1}) \right. \\ & \quad \left. + \frac{3}{2}\delta^{(cd)}(\xi^e \cdot \alpha_{-1}) \left[\frac{1}{4}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) + \frac{1}{2}(\delta \cdot \alpha_{-2}) \right] \right. \\ & \quad \left. + 3[(\eta^b \cdot \mathbf{r})(\zeta^a \cdot \xi^{(c)} - (\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^{(c)}))](\xi^d \cdot \alpha_{-1})(\xi^e \cdot \alpha_{-1}) \right. \\ & \quad \left. + \frac{3}{2}\delta^{(cd)}[(\xi^e \cdot \zeta^a)(\eta^b \cdot \mathbf{r}) - (\xi^e \cdot \eta^b)(\zeta^a \cdot \mathbf{v})] \left[\frac{1}{4}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) + \frac{1}{2}(\delta \cdot \alpha_{-2}) \right] \right. \\ & \quad \left. + 6(\zeta^a \cdot \xi^{(c)})(\eta^{[b]} \cdot \xi^d)(\xi^e \cdot \alpha_{-1}) \right\} |\mathbf{v} + \frac{3}{2}\delta\rangle \end{aligned} \quad (5.75)$$

for (5.65). When we insert the above two expressions into formula (5.73) the remaining oscillators on leg 1 can be removed by feeding them back again onto the other two legs. A glimpse at the overlaps shows that the result is very simple because we only have groundstates on the other two legs. This means that $\alpha_{-m}^{(1)}$ just becomes $-\alpha_0^{(3)}$. Consequently, we can replace $\delta \cdot \alpha_{-m}^{(1)}$ by -1 and $\xi^c \cdot \alpha_{-m}$ by $\xi^c \cdot \mathbf{s}$, respectively. Let us now consider the following linear combination of the above states:

$$\psi := \left(FA_{-3}^c(\mathbf{v}) + GA_{-1}^c(\mathbf{v})A_{-1}^c(\mathbf{v})A_{-1}^c(\mathbf{v}) + HA_{-1}^c(\mathbf{v}) \sum_{d=1}^8 A_{-1}^d(\mathbf{v})A_{-1}^d(\mathbf{v}) \right) |\mathbf{v}\rangle. \quad (5.76)$$

If we put this on leg 1 in (5.73) and take into account the preceding calculations we finally get

$$\begin{aligned} & V^{\text{CSV}} \left\{ |\psi\rangle_1 \otimes (\zeta^a \cdot \alpha_{-1}^{(2)})| - \mathbf{r} - \delta \rangle_2 \otimes (\eta^b \cdot \alpha_{-1}^{(3)})| - \mathbf{s} - \delta \rangle_3 \right\} \\ &= (\xi^c \cdot \mathbf{s}) [(\zeta^a \cdot \eta^b) - (\eta^b \cdot \mathbf{r})(\zeta^a \cdot \mathbf{v})] \left\{ -\frac{1}{8}F + G[(\xi^c \cdot \mathbf{s})^2 - \frac{3}{8}] + H \left(\sum_{d=1}^8 (\xi^d \cdot \mathbf{s})^2 - \frac{5}{4} \right) \right\} \\ & \quad + [(\zeta^a \cdot \xi^c)(\eta^b \cdot \mathbf{r}) - (\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^c)] \left\{ \frac{3}{8}F + 3G[(\xi^c \cdot \mathbf{s})^2 - \frac{1}{8}] + H \left(\sum_{d=1}^8 (\xi^d \cdot \mathbf{s})^2 - \frac{5}{4} \right) \right\} \\ & \quad + 6G(\zeta^a \cdot \xi^c)(\eta^b \cdot \xi^c)(\xi^c \cdot \mathbf{s}) + 2H(\xi^c \cdot \mathbf{s}) \sum_{d=1}^8 (\xi^d \cdot \mathbf{s}) [(\zeta^a \cdot \xi^d)(\eta^b \cdot \mathbf{r}) - (\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^d)] \\ & \quad + 2H \sum_{d=1}^8 [(\zeta^a \cdot \xi^c)(\eta^b \cdot \xi^d)(\xi^d \cdot \mathbf{s}) + (\zeta^a \cdot \xi^d)(\eta^b \cdot \xi^c)(\xi^d \cdot \mathbf{s}) + (\zeta^a \cdot \xi^d)(\eta^b \cdot \xi^d)(\xi^c \cdot \mathbf{s})]. \end{aligned} \quad (5.77)$$

Note that the momenta were just right for (5.72) to give 1 on the right-hand side. For decoupling the above expression has to vanish for any choice of ζ^a and η^b . This analysis has to be performed case by case.

(1) Let $\xi^c \cdot \mathbf{s} = 0$, i.e. $c = 1, \dots, 7$. Then the right-hand side reduces to

$$[(\zeta^a \cdot \xi^c)(\eta^b \cdot \mathbf{r}) - (\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^c)] \left[\frac{3}{8}F - \frac{3}{8}G - \frac{1}{4}H \right] + 2H \sum_{d=1}^8 (\xi^d \cdot \mathbf{s}) [(\zeta^a \cdot \xi^c)(\eta^b \cdot \xi^d) + (\zeta^a \cdot \xi^d)(\eta^b \cdot \xi^c)].$$

- (i) If $a = b = 8$ or both $a \neq 8$ and $b \neq 8$ then this expression identically vanishes.
- (ii) If either $a = 8$ or $b = 8$ then we are left with the relation

$$F - G - \frac{10}{3}H = 0. \quad (5.78)$$

(2) Let $\xi^c \cdot \mathbf{s} = -1$, i.e. $c = 8$. Then the right-hand side simplifies to

$$\begin{aligned} & [(\zeta^a \cdot \eta^b) - (\eta^b \cdot \mathbf{r})(\zeta^a \cdot \mathbf{v})] \left[\frac{1}{8}F - \frac{5}{8}G + \frac{1}{4}H \right] \\ & + [(\zeta^a \cdot \xi^c)(\eta^b \cdot \mathbf{r}) - (\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^c)] \left[\frac{3}{8}F + \frac{21}{8}G - \frac{1}{4}H \right] \\ & + 2H \sum_{d=1}^8 (\xi^d \cdot \mathbf{s}) [- (\zeta^a \cdot \xi^d)(\eta^b \cdot \mathbf{r}) + (\zeta^a \cdot \mathbf{v})(\eta^b \cdot \xi^d) + (\zeta^a \cdot \xi^c)(\eta^b \cdot \xi^d) + (\zeta^a \cdot \xi^d)(\eta^b \cdot \xi^c)] \\ & - 6G(\zeta^a \cdot \xi^c)(\eta^b \cdot \xi^c) - 2H \sum_{d=1}^8 (\zeta^a \cdot \xi^d)(\eta^b \cdot \xi^d), \end{aligned} \quad (5.79)$$

where we have also used $\sum_{d=1}^8 (\xi^d \cdot \mathbf{s}) = -1$ and $\sum_{d=1}^8 (\xi^d \cdot \mathbf{s})^2 = 1$.

- (i) If both $a \neq 8$ and $b \neq 8$ then only the first term and the last term contribute and we get the condition

$$F - 5G - 14H = 0. \quad (5.80)$$

- (ii) If either $a = 8$ or $b = 8$ then the expression identically vanishes.

- (iii) If $a = b = 8$ then all terms are nonzero and we find after insertion of the various scalar products

$$17F + 11G + 18H = 0. \quad (5.81)$$

Note that the last condition is just $24 \times (5.78) - 7 \times (5.80)$, i.e. the three equations are linearly dependent and only two of three parameters F, G, H are determined. If we put $F = 2$ then it is easy to check that we get $G = -8$ and $H = 3$. Hence we have verified that the linear combinations appearing in (5.32) indeed decouple in the expression (5.66).

It remains to show that this linear combination also decouples in (5.67). The calculation is completely analogous to the one above. Putting the Chevalley involuted version of (5.63) on leg 2 and using the oscillator overlaps we get

$$\begin{aligned} & V^{\text{CSV}} \left\{ |\psi\rangle_1 \otimes [- (\zeta^a \cdot \alpha_{-2}^{(2)}) + 2(\zeta^a \cdot \alpha_{-1}^{(2)})(\delta \cdot \alpha_{-1}^{(2)})] | - \mathbf{r} - 2\delta \rangle_2 \otimes | - \mathbf{s} \rangle_3 \right\} \\ & = V^{\text{CSV}} \left\{ [\zeta^a \cdot \alpha_0^{(1)} - 2\zeta^a \cdot \alpha_1^{(1)} + \zeta^a \cdot \alpha_2^{(1)} \right. \\ & \quad \left. + 2(\zeta^a \cdot \alpha_1^{(1)} - \zeta^a \cdot \alpha_0^{(1)})(\delta \cdot \alpha_1^{(1)} - \delta \cdot \alpha_0^{(1)})] |\psi\rangle_1 \otimes | - \mathbf{r} - 2\delta \rangle_2 \otimes | - \mathbf{s} \rangle_3 \right\}. \end{aligned} \quad (5.82)$$

Explicitly, we obtain

$$\begin{aligned} & [\zeta^a \cdot \alpha_0^{(1)} - 2\zeta^a \cdot \alpha_1^{(1)} + \zeta^a \cdot \alpha_2^{(1)} + 2(\zeta^a \cdot \alpha_1^{(1)} - \zeta^a \cdot \alpha_0^{(1)})(\delta \cdot \alpha_1^{(1)} - \delta \cdot \alpha_0^{(1)})] A_{-3}^c(\mathbf{v}) |\mathbf{v}\rangle \\ & = [- 3\zeta^a \cdot \alpha_0^{(1)} + 2\zeta^a \cdot \alpha_1^{(1)} + \zeta^a \cdot \alpha_2^{(1)}] A_{-3}^c(\mathbf{v}) |\mathbf{v}\rangle \\ & = \left\{ - 3(\zeta^a \cdot \mathbf{v}) [(\xi^c \cdot \alpha_{-3}) + \frac{3}{2}(\xi^c \cdot \alpha_{-2})(\delta \cdot \alpha_{-1}) + (\xi^c \cdot \alpha_{-1}) [\frac{3}{4}(\delta \cdot \alpha_{-2}) + \frac{9}{8}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1})] \right. \\ & \quad \left. + (\zeta^a \cdot \xi^c) [\frac{3}{2}(\delta \cdot \alpha_{-2}) + \frac{9}{4}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) + 3(\delta \cdot \alpha_{-1})] \right\} |\mathbf{v} + \frac{3}{2}\delta\rangle \end{aligned} \quad (5.83)$$

for (5.64) and

$$\begin{aligned} & [\zeta^a \cdot \alpha_0^{(1)} - 2\zeta^a \cdot \alpha_1^{(1)} + \zeta^a \cdot \alpha_2^{(1)} + 2(\zeta^a \cdot \alpha_1^{(1)} - \zeta^a \cdot \alpha_0^{(1)})(\delta \cdot \alpha_1^{(1)} - \delta \cdot \alpha_0^{(1)})] A_{-1}^c(\mathbf{v}) A_{-1}^d(\mathbf{v}) A_{-1}^e(\mathbf{v}) |\mathbf{v}\rangle \\ & = [- 3\zeta^a \cdot \alpha_0^{(1)} + 2\zeta^a \cdot \alpha_1^{(1)} + \zeta^a \cdot \alpha_2^{(1)}] A_{-1}^c(\mathbf{v}) A_{-1}^d(\mathbf{v}) A_{-1}^e(\mathbf{v}) |\mathbf{v}\rangle \\ & = \left\{ - 3(\zeta^a \cdot \mathbf{v}) [(\xi^c \cdot \alpha_{-1})(\xi^d \cdot \alpha_{-1})(\xi^e \cdot \alpha_{-1}) + \frac{3}{2}\delta^{cd}(\xi^e \cdot \alpha_{-1}) [\frac{1}{4}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) + \frac{1}{2}(\delta \cdot \alpha_{-2})] \right. \\ & \quad \left. + 6(\zeta^a \cdot \xi^c)(\xi^d \cdot \alpha_{-1})(\xi^e \cdot \alpha_{-1}) + 3\delta^{cd}(\xi^e \cdot \zeta^a) [\frac{1}{4}(\delta \cdot \alpha_{-1})(\delta \cdot \alpha_{-1}) + \frac{1}{2}(\delta \cdot \alpha_{-2})] \right\} |\mathbf{v} + \frac{3}{2}\delta\rangle \end{aligned} \quad (5.84)$$

for (5.65). When we insert the above two expressions into formula (5.82) the remaining oscillators on leg 1 can be removed by feeding them back again onto the other two legs. This amounts to replacing $\delta \cdot \alpha_{-m}$ by -1 and $\xi^c \cdot \alpha_{-m}$ by $\xi^c \cdot \mathbf{s}$, respectively. If we put the linear combination (5.76) on leg 1 in (5.82) and take into account the preceding calculations we finally get

$$\begin{aligned}
V^{\text{CSV}} & \left\{ |\psi\rangle_1 \otimes [- (\zeta^a \cdot \alpha_{-2}^{(2)}) + 2(\zeta^a \cdot \alpha_{-1}^{(2)})(\delta \cdot \alpha_{-1}^{(2)})] | - \mathbf{r} - 2\delta\rangle_2 \otimes | - \mathbf{s}\rangle_3 \right\} \\
& = -3(\xi^c \cdot \mathbf{s})(\zeta^a \cdot \mathbf{v}) \left\{ -\frac{1}{8}F + G[(\xi^c \cdot \mathbf{s})^2 - \frac{3}{8}] + H \left(\sum_{d=1}^8 (\xi^d \cdot \mathbf{s})^2 - \frac{5}{4} \right) \right\} \\
& \quad + 2(\zeta^a \cdot \xi^c) \left\{ -\frac{9}{8}F + 3G[(\xi^c \cdot \mathbf{s})^2 - \frac{1}{8}] + H \left(\sum_{d=1}^8 (\xi^d \cdot \mathbf{s})^2 - \frac{5}{4} \right) \right\} \\
& \quad + 4H(\xi^c \cdot \mathbf{s}) \sum_{d=1}^8 (\xi^d \cdot \mathbf{s})(\zeta^a \cdot \xi^d). \tag{5.85}
\end{aligned}$$

This decoupling analysis again has to be performed case by case.

(1) Let $\xi^c \cdot \mathbf{s} = 0$, i.e. $c = 1, \dots, 7$. Then the right-hand side reduces to

$$2(\zeta^a \cdot \xi^c) \left[-\frac{9}{8}F - \frac{3}{8}G - \frac{1}{4}H \right].$$

(i) If $a = 8$ then this expression identically vanishes.

(ii) If $a \neq 8$ then we are left with the relation

$$3F + G + \frac{2}{3}H = 0. \tag{5.86}$$

(2) Let $\xi^c \cdot \mathbf{s} = -1$, i.e. $c = 8$. Then the right-hand side simplifies to

$$3(\zeta^a \cdot \mathbf{v}) \left[-\frac{1}{8}F + \frac{5}{8}G - \frac{1}{4}H \right] + 2(\zeta^a \cdot \xi^c) \left[-\frac{9}{8}F + \frac{21}{8}G - \frac{1}{4}H \right] - 4H \sum_{d=1}^8 (\xi^d \cdot \mathbf{s})(\zeta^a \cdot \xi^d). \tag{5.87}$$

(i) If $a = 8$ we get the condition

$$F - G - \frac{10}{3}H = 0. \tag{5.88}$$

(ii) If $a \neq 8$ then the expression identically vanishes.

We observe that Eq. (5.78) and Eq. (5.88) are identical and that Eq. (5.86) can be written as $4 \times (5.78) - (5.80)$. This means that the last two equations are compatible with the previous ones and hence we have verified that the linear combinations appearing in (5.32) are indeed not elements of the root space $E_{10}^{(\Lambda_7)}$.

This result is remarkable in several ways. Firstly, it exemplifies an as yet ill understood mechanism by which certain transversal physical states decouple from other transversal ones. Secondly, the calculation crucially depends on properties of the root lattice; this means that the result will be completely different when we perform the analysis for another lattice. Finally, we should stress again that, by this method, we have found eight states orthogonal to the root space $E_{10}^{(\Lambda_7)}$ without explicitly computing a single commutator! All this makes transversal decoupling even more miraculous a phenomenon than longitudinal decoupling.

A The measure

We consider an infinitesimal shift of a fixed single Koba Nielsen point, z_i , say:

$$\tilde{z}_j := z_j + \delta_{ij}\epsilon \quad \forall j. \tag{A.1}$$

If this shift is implemented by conformal transformations \mathcal{M}_j acting on the N -vertex it then follows that

$$-\frac{\partial}{\partial z_i} V^{[N]}(\{z_j\}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [V^{[N]}(\{z_j\}) - \tilde{V}^{[N]}(\{\tilde{z}_j\})] = V^{[N]}(\{z_j\}) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[1 - \prod_{j=1}^N \mathcal{M}_j^{(j)} \right]. \tag{A.2}$$

For the scattering amplitudes Eq. (3.1) this implies

$$\begin{aligned}
& \oint \frac{dz_1}{2\pi i} \dots \oint \frac{dz_N}{2\pi i} \Delta(z_1, z_2, z_3) \frac{\partial}{\partial z_i} \bar{\mu}(\{z_j\}) V^{[N]}(\{z_j\}) |\psi_1\rangle_1 \dots |\psi_N\rangle_N \\
&= - \oint \frac{dz_1}{2\pi i} \dots \oint \frac{dz_N}{2\pi i} \Delta(z_1, z_2, z_3) \bar{\mu}(\{z_j\}) \frac{\partial}{\partial z_i} V^{[N]}(\{z_j\}) |\psi_1\rangle_1 \dots |\psi_N\rangle_N \\
&= \oint \frac{dz_1}{2\pi i} \dots \oint \frac{dz_N}{2\pi i} \Delta(z_1, z_2, z_3) \bar{\mu}(\{z_j\}) V^{[N]}(\{z_j\}) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\mathbf{1} - \prod_{j=1}^N \hat{\mathcal{M}}_j^{(j)} \right] |\psi_1\rangle_1 \dots |\psi_N\rangle_N, \quad (\text{A.3})
\end{aligned}$$

where

$$\Delta(z_1, z_2, z_3) := \frac{(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)}{(z_1 - z_1^{(0)})(z_2 - z_2^{(0)})(z_3 - z_3^{(0)})}. \quad (\text{A.4})$$

This means that, once we know the operators $\hat{\mathcal{M}}_j$, we can determine the function $\bar{\mu}$ by solving the differential equation

$$\frac{\partial}{\partial z_i} \bar{\mu}(\{z_j\}) = \bar{\mu}(\{z_j\}) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\mathbf{1} - \prod_{j=1}^N \hat{\mathcal{M}}_j^{(j)} \right], \quad (\text{A.5})$$

which is to be understood as a relation between operators acting on arbitrary states.

Let us now determine the conformal transformations implementing the above shift of the Koba Nielsen point z_i . We have

$$\tilde{\xi}_j(\zeta) = [\mathcal{M}_j^{-1} \circ \xi_j](\zeta), \quad (\text{A.6})$$

i.e.,

$$\mathcal{M}_j = \xi_j \tilde{\xi}_j^{-1}. \quad (\text{A.7})$$

Since we are only interested in infinitesimal changes of the Koba Nielsen points we consider the expansion of ξ_j and $\tilde{\xi}_j$ around z_j and \tilde{z}_j , respectively:

$$\xi_j(\zeta) = \xi'_j(\zeta - z_j) + \mathcal{O}[(\zeta - z_j)^2], \quad (\text{A.8})$$

$$\tilde{\xi}_j(\zeta) = \tilde{\xi}'_j(\zeta - \tilde{z}_j) + \mathcal{O}[(\zeta - \tilde{z}_j)^2], \quad (\text{A.9})$$

where

$$\xi'_j \equiv \left. \frac{d\xi_j}{d\zeta} \right|_{\zeta=z_j}, \quad \tilde{\xi}'_j \equiv \left. \frac{d\tilde{\xi}_j}{d\zeta} \right|_{\zeta=\tilde{z}_j}. \quad (\text{A.10})$$

Inverting the above expansion for $\tilde{\xi}_j(\zeta)$ we get

$$\tilde{\xi}_j^{-1}(\zeta) \approx \tilde{z}_j + \frac{1}{\tilde{\xi}'_j} \zeta. \quad (\text{A.11})$$

Hence the conformal transformations associated with the infinitesimal shift of the Koba Nielsen point z_i have the form

$$\mathcal{M}_j \approx \xi'_j \left(\tilde{z}_j - z_j + \frac{1}{\tilde{\xi}'_j} \zeta \right) = \epsilon \delta_{ij} \xi'_j + \frac{\xi'_j}{\tilde{\xi}'_j} \zeta, \quad (\text{A.12})$$

or, as operators,

$$\begin{aligned}
\hat{\mathcal{M}}_j &\approx e^{\epsilon \delta_{ij} \xi'_j L_{-1}} \left(\frac{\xi'_j}{\tilde{\xi}'_j} \right)^{L_0} \\
&\approx [\mathbf{1} + \epsilon \delta_{ij} \xi'_j L_{-1}] \left(\frac{\xi'_j}{\tilde{\xi}'_j} \right)^{L_0}. \quad (\text{A.13})
\end{aligned}$$

Suppose now that the states ψ_j for $j \neq i$ occurring in (A.3) are all physical and that $\psi_i \equiv \Omega \in \mathcal{P}^0$, i.e. $L_n \Omega = 0 \forall n \geq 0$. It then follows that

$$\begin{aligned} & \prod_{j=1}^N \hat{\mathcal{M}}_j^{(j)} |\psi_1\rangle_1 \dots |\psi_{i-1}\rangle_{i-1} |\Omega\rangle_i |\psi_{i+1}\rangle_{i+1} \dots |\psi_N\rangle_N \\ &= \prod_{\substack{j=1 \\ j \neq i}}^N \left(\frac{\xi'_j}{\tilde{\xi}'_j} \right) \left[\mathbf{1} + \epsilon \xi'_i L_{-1}^{(i)} \right] |\psi_1\rangle_1 \dots |\psi_{i-1}\rangle_{i-1} |\Omega\rangle_i |\psi_{i+1}\rangle_{i+1} \dots |\psi_N\rangle_N. \end{aligned} \quad (\text{A.14})$$

Obviously $L_{-1}^{(i)} |\Omega\rangle_i$ is a null physical state and will not contribute to a physical string scattering amplitude (cf. Def. 3):

$$\oint \frac{dz_1}{2\pi i} \dots \oint \frac{dz_N}{2\pi i} \Delta(z_1, z_2, z_3) \mu(\{z_j\}) V^{[N]}(\{z_j\}) L_{-1}^{(i)} |\Omega\rangle_i \prod_{\substack{j=1 \\ j \neq i}}^N |\psi_j\rangle_j = 0, \quad (\text{A.15})$$

where $\mu \equiv \xi'_i \bar{\mu}$ denotes the true measure determined by decoupling. Consequently, we are left with the following differential equation for the measure:

$$\begin{aligned} \frac{\partial}{\partial z_i} \left(\frac{\mu}{\xi'_i} \right) &= \frac{\mu}{\xi'_i} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\mathbf{1} - \prod_{\substack{j=1 \\ j \neq i}}^N \frac{\xi'_j}{\tilde{\xi}'_j} \right] \\ &= \frac{\mu}{\xi'_i} \frac{\partial}{\partial z_i} \log \left[\prod_{\substack{j=1 \\ j \neq i}}^N \xi'_j \right], \end{aligned} \quad (\text{A.16})$$

which has the solution (up to multiplication by a constant)

$$\mu = \prod_{j=1}^N \xi'_j = \prod_{j=1}^N \left. \frac{\partial \xi_j}{\partial \zeta} \right|_{\zeta=z_j}. \quad (\text{A.17})$$

If we choose ξ_k , say, as the coordinate ζ , then the measure takes the form

$$\mu = \prod_{j=1}^N \left. \frac{\partial \tau_{jk}}{\partial \xi_k} \right|_{\xi_j=0}. \quad (\text{A.18})$$

Let us work out some examples.

(i) For the choice [41]

$$\xi_j := \frac{(\zeta - z_j)(z_{j+1} - z_{j-1})}{(\zeta - z_{j-1})(z_{j+1} - z_j)}, \quad (\text{A.19})$$

or,

$$\tau_{ji}(\xi_i) := \frac{[(z_{i+1} - z_i)(z_{i-1} - z_j)\xi_i + (z_{i+1} - z_{i-1})(z_j - z_i)](z_{j+1} - z_{j-1})}{[(z_{i+1} - z_i)(z_{i-1} - z_{j-1})\xi_i + (z_{i+1} - z_{i-1})(z_{j-1} - z_i)](z_{j+1} - z_j)}, \quad (\text{A.20})$$

we find the measure to be

$$\mu = \prod_{j=1}^N \frac{z_{j+1} - z_{j-1}}{(z_{j+1} - z_j)(z_j - z_{j-1})}, \quad (\text{A.21})$$

while the zero mode term (3.13) after some algebra is given by

$$\mathcal{N} = \prod_{i < j} \left[- \frac{(z_{j+1} - z_j)(z_j - z_{j-1})(z_{i+1} - z_i)(z_i - z_{i-1})}{(z_{j+1} - z_{j-1})(z_{i+1} - z_{i-1})(z_j - z_i)^2} \right]^{-\frac{1}{2} \mathbf{p}_i \cdot \mathbf{p}_j}. \quad (\text{A.22})$$

When we rearrange this product and use momentum conservation we arrive at the formula

$$\mathcal{N} = \prod_{j=1}^N \left[\frac{(z_{j+1} - z_j)(z_j - z_{j-1})}{z_{j+1} - z_{j-1}} \right]^{\frac{1}{2}\mathbf{p}_j^2} \prod_{i<j} (z_i - z_j)^{\mathbf{p}_i \cdot \mathbf{p}_j}. \quad (\text{A.23})$$

Note that for tachyon scattering, i.e. $\mathbf{p}_j^2 = 2 \forall j$, the measure is cancelled by the first product in the zero mode term.

(ii) For the simple choice:

$$\xi_j(\zeta) := \zeta - z_j, \quad (\text{A.24})$$

or,

$$\tau_{ji}(\xi_i) := \xi_i + z_i - z_j, \quad (\text{A.25})$$

the measure comes out to be a constant,

$$\mu = 1, \quad (\text{A.26})$$

while the zero mode term (3.13) is given by

$$\mathcal{N} = \prod_{i<j} (z_i - z_j)^{\mathbf{p}_i \cdot \mathbf{p}_j}. \quad (\text{A.27})$$

(iii) For the “vertex operator choice”:

$$\tau_{12} = \Gamma, \quad \tau_{1i} = \frac{1}{\xi_i + z_i}, \quad \tau_{i1} = \frac{1}{\xi_1} - z_i, \quad \tau_{ij} = \xi_j + z_j - z_i, \quad (\text{A.28})$$

we may take $\xi_2 \equiv \zeta$ so that $z_2 = 0$, $z_1 = \infty$ and the measure becomes

$$\mu = -\frac{1}{z_1^2}, \quad (\text{A.29})$$

which is cancelled by the Faddeev Popov determinant (for $z_2 = 0$, $z_1 \rightarrow \infty$)

$$(z_1 - z_2)(z_2 - z_3)(z_1 - z_3) = (-z_1^2)z_3, \quad (\text{A.30})$$

to give a finite result as $z_1 \rightarrow \infty$. Evaluation of the zero mode term finally yields

$$\mathcal{N} = \prod_{2 \leq i < j \leq N} (z_i - z_j)^{\mathbf{p}_i \cdot \mathbf{p}_j}. \quad (\text{A.31})$$

For tachyon scattering, i.e. $\mathbf{p}_j^2 = 2 \forall j$, we observe that the product $\mu\mathcal{N}$ of the measure and the zero mode term always reproduces the well-known Koba Nielsen formula.

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